M2S1 - ASSESSED COURSEWORK 2, 2011-12

The submission deadline is Monday 12th December, 2.00pm.
Please hand in to the Undergraduate Mathematics Student Office

1. Let Z be a standard normal random variable. Show that

$$\frac{1}{\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2} > P(Z > t) > \frac{1}{\sqrt{2\pi}}\frac{t}{t^2+1}e^{-t^2/2}.$$

[*Hint.* For the right hand inequality, you might wish to consider the function

$$g(t) = P(Z > t) - \frac{1}{\sqrt{2\pi}}\frac{t}{t^2+1}e^{-t^2/2},$$

and show that this is strictly decreasing.] *[6 MARKS]*

2. Let X have the Gamma$(s, 1)$ distribution. Given that $X = x$, let Y have the Poisson distribution with parameter x.

Find the moment generating function of Y and show that

$$\frac{Y - E(Y))}{\sqrt{\text{var}(Y)}} \xrightarrow{d} W, \text{ as } s \to \infty,$$

for a random variable W which you should identify. *[7 MARKS]*

3. Let X and Y be independent standard normal random variables.

Find the distributions of: (a) $X/|Y|$; (b) $X/(X+Y)$.

[*Hint.* A random variable Z has the Cauchy(μ, σ) distribution if it has probability density function of the form

$$f(z;\mu,\sigma) = \frac{1}{\pi\sigma[1+(\frac{z-\mu}{\sigma})^2]},\ z \in \mathbb{R}.$$

Note that $Z = (Z^{-1})^{-1}$.] *[7 MARKS]*

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1. First, $P(Z > t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$$< \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-\frac{1}{2}x^2} dx, \text{ since } \frac{x}{t} > 1 \text{ if } x > t$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{t} \left[-e^{-\frac{1}{2}x^2} \right]_t^\infty = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{t} e^{-\frac{1}{2}t^2}$$

2 marks

Let $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$ be the pdf of $N(0,1)$ and $\Phi(t)$ be the corresponding cdf.

Note that $\phi'(t) = -t\phi(t)$ and define

$$g(t) = 1 - \Phi(t) - \frac{t}{t^2+1}\phi(t).$$

Then $g(0) > 0$ and $g(t) \longrightarrow 0$ as

$t \to \infty$. Also, differentiation gives [note, $\bar{\Phi}'(t) = \phi(t)$]

$$g'(t) = -\frac{2\phi(t)}{(1+t^2)^2} < 0,$$

so g is strictly decreasing. Combined with above properties, we conclude that g must always be positive, proving the right hand inequality.

[<u>4 marks</u>. Marks awarded for any convincing and correct argument.]

2. The moment generating function of Y is
$$M_Y(t) = E(e^{tY})$$
$$= E[E[e^{tY} | X]]$$
$$= E[\exp(X(e^t - 1))],$$

Since, given $X = x$, Y is Poisson (x).

So, $M_Y(t) = M_X(e^t - 1)$

$$= \left(\frac{1}{1-(e^t-1)}\right)^s,$$

since X is Gamma $(s, 1)$.

⌜2 marks⌟

Directly, $E(Y) = E(E(Y|X)) = E(X) = s$,

$E(Y^2) = E(E(Y^2|X)) = E(X + X^2)$

$= s + \text{var}(X) + (E(X))^2 = s + s + s^2$

$= 2s + s^2$, so that $\text{var}(Y) = 2s + s^2 - s^2$

$= 2s$.

⌜or, calculate by differentiating the mgf⌟

⌜1 mark⌟

Let $Z = \dfrac{Y - E(Y)}{\sqrt{\text{var}(Y)}} = \dfrac{Y - s}{\sqrt{2s}}$.

This has mgf

$$M_Z(t) = E\left[e^{t(Y-s)/\sqrt{2s}}\right]$$

$$= e^{-t\sqrt{\frac{s}{2}}}\, E\left(e^{tY/\sqrt{2s}}\right)$$

$$= e^{-t\sqrt{\frac{s}{2}}}\, M_Y\left(\frac{t}{\sqrt{2s}}\right).$$

Then, $\log M_Y\left(\frac{t}{\sqrt{2s}}\right)$

$$= -s\log\left(2 - e^{t/\sqrt{2s}}\right)$$

$$= -s\log\left\{1 + \left(1 - e^{t/\sqrt{2s}}\right)\right\}$$

$$= -s\left\{\left(1 - e^{-t/\sqrt{2s}}\right) - \frac{1}{2}\left(1 - e^{-t/\sqrt{2s}}\right)^2 + \dots\right\},$$

[using $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$]

$$= -s\left\{\left(1-\left(1+\frac{t}{\sqrt{2s}}+\frac{t^2}{4s}+\dots\right)\right.\right.$$

$$\left.-\frac{1}{2}\left(-\frac{t}{\sqrt{2s}}-\frac{t^2}{4s}-\dots\right)^2+\dots\right\}$$

[using $e^x = 1 + x + \frac{x^2}{2} + \dots$]

$$= t\sqrt{\frac{s}{2}} + \frac{t^2}{4} + \frac{t^2}{4} + o(1),$$

where the $o(1)$ ['negligible in the limit'] terms are as $s \to \infty$.

So, $\log M_Z(t) = -t\sqrt{\frac{s}{2}} + t\sqrt{\frac{s}{2}}$

$$+\frac{t^2}{2} + o(1), \text{ as } s \to \infty,$$

$$M_Z(t) \to e^{\frac{1}{2}t^2} \text{ as } s \to \infty,$$

$$Z \xrightarrow{d} W \text{ as } n \to \infty, \underline{W \sim N(0,1)}$$

⌜ 4 marks ⌟

Again, marks awarded for any argument that convinces the marker. Full marks requires explicit identification of $W \sim N(0,1)$.

3. (a) $$\frac{X}{|Y|} \equiv \frac{X}{\sqrt{Y^2/1}} \sim t_1$$
$$\equiv \underline{\text{Cauchy}(0,1)},$$

by definition of t-distribution.

⌜ 2 marks. OK if say t_1 ⌟.

(b). ⌜ Various potential routes here, using, for example, multivariate transformation. Marks to be given for any correct derivation. ⌟

Let $Z = \dfrac{X}{X+Y}$. Then $U = Z^{-1} = 1 + \dfrac{Y}{X}$

$\equiv 1 + W$, W is Cauchy$(0,1)$.

Univariate transformation gives that U has pdf
$$f_U(u) = \frac{1}{\pi\left(1+(u-1)^2\right)}, \quad u \in \mathbb{R}$$

Then let $Z = U^{-1}$. The Jacobian is

$J(z) = -\frac{1}{z^2}$ and $f_Z(z) = f_U\left(\frac{1}{z}\right)|J(z)|$

$$= \frac{1}{\pi\left(1+\left(\frac{1}{z}-1\right)^2\right)} \cdot \frac{1}{z^2} = \frac{1}{\pi \cdot \frac{1}{2}\left(1+\frac{\left(z-\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2}\right)},$$

on simplification. So $Z \sim \underline{\text{Cauchy}\left(\frac{1}{2}, \frac{1}{2}\right)}$

⌜ <u>5 marks</u> ⌟

⌜Deduct 2 marks if Cauchy parameters $\mu = \sigma = \frac{1}{2}$ not stated correctly.⌟