DISTRIBUTIONS FACTSHEET

DISCRETE DISTRIBUTIONS

Models based on an independent sequence of identical 'Bernoulli' trials with success probability θ.

- **BERNOULLI** X is the total number of successes in one trial.
- **BINOMIAL** X is the total number of successes in n trials.
- **GEOMETRIC** X is the total number of trials required to obtain **one** success.
- **NEGATIVE BINOMIAL** X is the total number of trials required to obtain **n** successes. Alternative form given by considering $Y = X - n$, to give a distribution on $\{0, 1, 2, \ldots\}$.
- **POISSON** X is the count of the number of events in a given (continuous) time interval. The Poisson distribution is obtained as the limiting form as $n \to \infty$ of the $Binomial(n, \theta)$ distribution, with $\lambda = n\theta$ held fixed.

Connections:

- Bernoulli/Binomial

$$X_1, \ldots, X_n \text{ IID } Bernoulli(\theta) \quad \Rightarrow \quad Y = \sum_{i=1}^{n} X_i \sim Binomial(n, \theta).$$

- Geometric/Negative Binomial

$$X_1, \ldots, X_n \text{ IID } Geometric(\theta) \quad \Rightarrow \quad Y = \sum_{i=1}^{n} X_i \sim NegBinomial(n, \theta).$$

- Binomial/Poisson

$$X_n \sim Binomial(n, \theta) \longrightarrow X \sim Poisson(\lambda),$$

  where $\lambda = n\theta$ is held fixed and $n \longrightarrow \infty$.

- Negative Binomial/Poisson

$$X_n \sim NegBinomial(n, \theta), \quad Y_n = X_n - n \longrightarrow Y \sim Poisson(\lambda),$$

  where $\lambda = n(1-\theta)/\theta$ is held fixed and $n \longrightarrow \infty$.

Summations of Independent RVs:

- Binomial

$$\left.\begin{array}{r} X \sim Binomial(m, \theta) \\ Y \sim Binomial(n, \theta) \end{array}\right\} \quad \Rightarrow \quad T = X + Y \sim Binomial(m + n, \theta)$$

- Negative Binomial

$$\left.\begin{array}{r} X \sim NegBinomial(m, \theta) \\ Y \sim NegBinomial(n, \theta) \end{array}\right\} \quad \Rightarrow \quad T = X + Y \sim NegBinomial(m + n, \theta)$$

- Poisson

$$\left.\begin{array}{r} X \sim Poisson(\lambda_X) \\ Y \sim Poisson(\lambda_Y) \end{array}\right\} \quad \Rightarrow \quad T = X + Y \sim Poisson(\lambda_X + \lambda_Y)$$

CONTINUOUS DISTRIBUTIONS

- Distributions on $\mathbb{R}^+$

  Begin with $U \sim Uniform(0,1)$:

  ▶ $X = -\frac{1}{\lambda} \log U \sim Exponential(\lambda)$, for $\lambda > 0$.

  ▶ $Y = X^{1/\alpha} \sim Weibull(\alpha, \lambda)$, for $\alpha > 0$.

  ▶ If $X_1, \ldots, X_n \sim Exponential(\lambda)$, independent, then $Y = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$.

  ▶ If $X \sim Gamma(\alpha_X, \lambda)$ and $Y \sim Gamma(\alpha_Y, \lambda)$ are independent, then

  $$T = X + Y \sim Gamma(\alpha_X + \alpha_Y, \lambda).$$

- Poisson Process links

  Consider events occurring independently at a constant rate λ in continuous time. Let

  $$\begin{aligned} X(t,s) &\equiv \text{number of events occurring in interval } [t,s) \\ X_i &\equiv \text{time between event } i-1 \text{ and event } i \\ Y_n &\equiv \text{time of event } n \end{aligned}$$

  ▶ $X(t,s) \sim Poisson(\lambda(s-t))$

  ▶ $X(0,t)$ and $X(t,s)$ are independent for $s > t$.

  ▶ $X_i \sim Exponential(\lambda)$, with $X_1, X_2, \ldots$ independent.

  ▶ $Y_n = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$.

- Distributions on $\mathbb{R}$: The Normal distribution and connections

  ▶ Suppose $X \sim N(0,1)$. Then $Y = \mu + \sigma X \sim N(\mu, \sigma^2)$.

  ▶ Suppose $X \sim N(0,1)$. Then $Y = X^2 \sim Gamma(1/2, 1/2) \equiv \chi^2_1$, the chisquared distribution on 1 degree of freedom.

  ▶ If $X_1, X_2 \sim N(0,1)$, and $V \sim \chi^2_\nu$ are all independent, then

  $$T_1 = \frac{X_1}{X_2} \sim Cauchy, \quad T_2 = \frac{X_1}{\sqrt{V/\nu}} \sim t_n,$$

  with t_n denoting Student's t-distribution on n degrees of freedom.

  ▶ If $V_1 \sim \chi^2_{\nu_1}$ and $V_2 \sim \chi^2_{\nu_2}$ are independent, then

  $$T_3 = \frac{V_1/\nu_1}{V_2/\nu_2} \sim F(\nu_1, \nu_2),$$

  the F-distribution on ν_1, ν_2 degrees of freedom.

- If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$Y = X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

- Distribution on $(0, 1)$: The Beta distribution
  - If $X_1 \sim Gamma(\alpha_1, \beta)$ and $X_2 \sim Gamma(\alpha_2, \beta)$ are independent, then

$$V = \frac{X_1}{X_1 + X_2} \sim Beta(\alpha_1, \alpha_2).$$