M2S1 - EXERCISES 8

Statistical Inference

1. Suppose that $X_1, ..., X_n$ are a random sample from a $Poisson(\lambda)$ distribution. Define statistics

$$T_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad T_2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Show, using properties of Poisson random variables and general properties of expectation and variance, that

$$E_{f_{T_1}}[T_1] = E_{f_{T_2}}[T_2] = \lambda.$$

2. Let S_n^2 denote the sample variance derived from a random sample of size n from $N(\mu, \sigma^2)$, so that

$$V_n = \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Show, using the Central Limit Theorem that

$$\frac{\sqrt{n-1}(S_n^2 - \sigma^2)}{\sigma^2\sqrt{2}} \xrightarrow{d} Z \sim N(0,1),$$

so that, for large n, S_n^2 is approximately distributed as $N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$.

3. Suppose that $X_1, ..., X_n$ are a random sample from a $Ga(\alpha, \beta)$ distribution. Find the method of moments estimators of α and β.

4. Find the *maximum likelihood estimators* of the unknown parameters in the following probability densities on the basis of a random sample of size n.

(i) $f_X(x;\theta) = \theta x^{\theta-1},\ 0 < x < 1, \theta > 0.$

(ii) $f_X(x;\theta) = (\theta+1)x^{-\theta-2},\ 1 < x, \theta > 0.$

(iii) $f_X(x;\theta) = \theta^2 x \exp\{-\theta x\},\ 0 < x, \theta > 0.$

(iv) $f_X(x;\theta) = 2\theta^2 x^{-3},\ \theta \le x, \theta > 0.$

(v) $f_X(x;\theta_1,\theta_2) = \theta_1\theta_2^{\theta_1}x^{-\theta_1-1},\ \theta_2 \le x,\ \ \theta_1, \theta_2 > 0.$

5. Recall, an estimator, T, is an *unbiased* estimator of the function $\tau(\theta)$ of a parameter θ if

$$\mathrm{E}_{f_T}[T] = \tau(\theta),$$

where f_T is the *sampling distribution* of T. The *bias*, $b(T)$, and *Mean Squared Error*, **MSE**, of an estimator T of $\tau(\theta)$ are defined respectively by

$$b(T) = \mathrm{E}_{f_T}[T] - \tau(\theta), \qquad MSE(T) = \mathrm{E}_{f_T}[(T - \tau(\theta))^2].$$

Suppose that $X_1, ..., X_n$ are a random sample from a $Poisson(\lambda)$ distribution. Find the maximum likelihood estimator of λ, and show that this estimator is unbiased. Also, find the maximum likelihood estimator of $\tau(\lambda) = e^{-\lambda}$ =P$[X = 0]$.

6. (i) Suppose that $X_1, ..., X_n$ are a random sample from the probability distribution with pdf

$$f_X(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \qquad x > 0.$$

Show that the sample mean $\bar{X}$ is an unbiased estimator of θ. Show also that, if the random variable Y_1 is defined as $Y_1 = \min\{X_1, ..., X_n\}$ then $Z = nY_1$ is also unbiased for θ.

(ii) Suppose now that $X_1, \ldots, X_n$ are a random sample from the probability distribution with pdf

$$f_X(x;\theta) = \theta e^{-\theta x}, \qquad x > 0.$$

Find the maximum likelihood estimator of θ and show that it is biased as an estimator of θ, but that some multiple of it is not.

Show that $2\theta \sum_{i=1}^{n} X_i$ is a pivotal quantity. Describe briefly how to use this to construct a $100(1-\alpha)\%$ confidence interval for θ, $\alpha \in (0,1)$. How would you test the null hypothesis $H_0 : \theta = \theta_0$, against the alternative $H_1 : \theta \neq \theta_0$?

7. Suppose that $X_1, ..., X_n$ are a random sample from the uniform distribution on $(\theta - 1, \theta + 1)$. Show that the sample mean $\bar{X}$ is an unbiased estimator of θ. Let Y_1 and Y_n be the smallest and largest order statistics derived from $X_1, ..., X_n$. Show also that random variable $M = (Y_1 + Y_n)/2$ is an unbiased estimator of θ.

8. Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$. Find the maximum likelihood estimator $\hat{\theta}$ of θ. By considering the distribution of $\hat{\theta}/\theta$ show that for $\alpha \in (0,1)$, a $100(1-\alpha)\%$ confidence interval for θ based on $\hat{\theta}$ is given by $(\hat{\theta}, \hat{\theta}/\alpha^{1/n})$.

9. Let X_1 be distributed as $N(\theta_1, 1)$, independently of X_2, distributed as $N(\theta_2, 1)$, with θ_1 and θ_2 both unknown. Show that both the square S and circle C given by

$$S = \{(\theta_1, \theta_2) : |X_1 - \theta_1| \leq 2.236, |X_2 - \theta_2| \leq 2.236\},$$
$$C = \{(\theta_1, \theta_2) : (X_1 - \theta_1)^2 + (X_2 - \theta_2)^2 \leq 5.991\}$$

are 95% confidence sets for (θ_1, θ_2). [*Hint*: $\Phi(2.236) = (1 + \sqrt{0.95})/2$, where Φ is the distribution function of $N(0,1)$]. What is a sensible criterion for choosing between S and C?

10. The following data is sampled from a $N(\mu, \sigma^2)$ distribution, where both μ and σ^2 are unknown: 6.82, 6.07, 3.74, 6.87, 5.92.

For this data $\sum x_i = 29.42, \sum x_i^2 = 179.588$.

(i) Find a 95% confidence interval for μ and show that its width is about 3.16.

(ii) Suppose it becomes known that the true value of σ^2 is 1. Show that a 95% confidence interval for μ now has width about 1.75.

Note that the width of the confidence interval has turned out to be narrower when the true value of σ^2 is known. Will this always happen?

(iii) [Harder]. Consider the event that the 95% confidence interval for μ is narrower when σ^2 is known than when it is unknown, given that both intervals are computed from the same random sample $X_1, \ldots, X_n$ from $N(\mu, \sigma^2)$. Show that for $n = 5$ this event has probability a bit less than 0.75. [*Hint:* you will need to refer to tables of the quantiles of the χ^2_4 distribution].