

## WORKED EXAMPLES 4

### 1-1 MULTIVARIATE TRANSFORMATIONS

Given a collection of variables  $(X_1, \dots, X_k)$  with range  $\mathbb{X}^{(k)}$  and joint pdf  $f_{X_1, \dots, X_k}$  we can construct the pdf of a transformed set of variables  $(Y_1, \dots, Y_k)$  using the following steps:

1. Write down the set of transformation functions  $g_1, \dots, g_k$

$$\begin{aligned} Y_1 &= g_1(X_1, \dots, X_k) \\ &\vdots \\ Y_k &= g_k(X_1, \dots, X_k) \end{aligned}$$

2. Write down the set of inverse transformation functions  $g_1^{-1}, \dots, g_k^{-1}$

$$\begin{aligned} X_1 &= g_1^{-1}(Y_1, \dots, Y_k) \\ &\vdots \\ X_k &= g_k^{-1}(Y_1, \dots, Y_k) \end{aligned}$$

3. Consider the joint range of the new variables,  $\mathbb{Y}^{(k)}$ .
4. Compute the Jacobian of the transformation: first form the matrix of partial derivatives

$$D_y = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_k} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_k}{\partial y_1} & \frac{\partial x_k}{\partial y_2} & \cdots & \frac{\partial x_k}{\partial y_k} \end{bmatrix}$$

where, for each  $(i, j)$

$$\frac{\partial x_i}{\partial y_j} = \frac{\partial}{\partial y_j} \{g_i^{-1}(y_1, \dots, y_k)\}$$

and then set  $|J(y_1, \dots, y_k)| = |\det D_y|$

*Note that*

$$\det D_y = \det D_y^T$$

*so that an alternative but equivalent Jacobian calculation can be carried out by forming  $D_y^T$ . Note also that*

$$|J(y_1, \dots, y_k)| = \frac{1}{|J(x_1, \dots, x_k)|}$$

*where  $J(x_1, \dots, x_k)$  is the Jacobian of the transformation regarded in the reverse direction (that is, if we start with  $(Y_1, \dots, Y_k)$  and transform to  $(X_1, \dots, X_k)$ ).*

5. Write down the joint pdf of  $(Y_1, \dots, Y_k)$  as

$$f_{Y_1, \dots, Y_k}(y_1, \dots, y_k) = f_{X_1, \dots, X_k}(g_1^{-1}(y_1, \dots, y_k), \dots, g_k^{-1}(y_1, \dots, y_k)) \times |J(y_1, \dots, y_k)|$$

for  $(y_1, \dots, y_k) \in \mathbb{Y}^{(k)}$ .

**EXAMPLE 1** Suppose that  $X_1$  and  $X_2$  have joint pdf

$$f_{X_1, X_2}(x_1, x_2) = 2, \quad 0 < x_1 < x_2 < 1,$$

and zero otherwise. Compute the joint pdf of random variables

$$Y_1 = \frac{X_1}{X_2}, \quad Y_2 = X_2.$$

**SOLUTION**

1. Given that  $\mathbb{X}^{(2)} \equiv \{(x_1, x_2) : 0 < x_1 < x_2 < 1\}$  and

$$g_1(t_1, t_2) = \frac{t_1}{t_2}, \quad g_2(t_1, t_2) = t_2.$$

2. Inverse transformations:

$$\left. \begin{array}{l} Y_1 = X_1/X_2 \\ Y_2 = X_2 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} X_1 = Y_1 Y_2 \\ X_2 = Y_2 \end{array} \right.$$

and thus

$$g_1^{-1}(t_1, t_2) = t_1 t_2, \quad g_2^{-1}(t_1, t_2) = t_2.$$

3. Range: to find  $\mathbb{Y}^{(2)}$  consider point by point transformation from  $\mathbb{X}^{(2)}$  to  $\mathbb{Y}^{(2)}$ . For a pair of points  $(x_1, x_2) \in \mathbb{X}^{(2)}$  and  $(y_1, y_2) \in \mathbb{Y}^{(2)}$  linked via the transformation, we have

$$0 < x_1 < x_2 < 1 \Leftrightarrow 0 < y_1 y_2 < y_2 < 1$$

and hence we can extract the inequalities

$$0 < y_2 < 1 \text{ and } 0 < y_1 < 1 \quad \therefore \quad \mathbb{Y}^{(2)} \equiv (0, 1) \times (0, 1).$$

4. The Jacobian for points  $(y_1, y_2) \in \mathbb{Y}^{(2)}$  is found from

$$D_y = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} \Rightarrow |J(y_1, y_2)| = |\det D_y| = |y_2| = y_2.$$

Note that for points  $(x_1, x_2) \in \mathbb{X}^{(2)}$  is

$$D_x = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{x_2} & \frac{x_1}{x_2^2} \\ 0 & 1 \end{bmatrix} \Rightarrow |J(x_1, x_2)| = |\det D_x| = \left| \frac{1}{x_2} \right| = \frac{1}{x_2},$$

verifying that

$$|J(y_1, y_2)| = \frac{1}{|J(x_1, x_2)|}.$$

5. Finally, we have

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1 y_2, y_2) \times y_2 = 2y_2, \quad 0 < y_1 < 1, 0 < y_2 < 1,$$

and zero otherwise.

**EXAMPLE 2** Suppose that  $X_1$  and  $X_2$  are **independent** and **identically distributed** random variables defined on  $\mathbb{R}^+$  each with pdf of the form

$$f_X(x) = \sqrt{\frac{1}{2\pi x}} \exp\left\{-\frac{x}{2}\right\}, \quad x > 0,$$

and zero otherwise. Compute the joint pdf of random variables  $Y_1 = X_1$  and  $Y_2 = X_1 + X_2$ .

**SOLUTION**

1. Given that  $\mathbb{X}^{(2)} \equiv \{(x_1, x_2) : 0 < x_1, 0 < x_2\}$  and

$$g_1(t_1, t_2) = t_1, \quad g_2(t_1, t_2) = t_1 + t_2.$$

2. Inverse transformations:

$$\left. \begin{array}{l} Y_1 = X_1 \\ Y_2 = X_1 + X_2 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} X_1 = Y_1 \\ X_2 = Y_2 - Y_1 \end{array} \right.$$

and thus

$$g_1^{-1}(t_1, t_2) = t_1, \quad g_2^{-1}(t_1, t_2) = t_2 - t_1.$$

3. Range: to find  $\mathbb{Y}^{(2)}$  consider point by point transformation from  $\mathbb{X}^{(2)}$  to  $\mathbb{Y}^{(2)}$ . For a pair of points  $(x_1, x_2) \in \mathbb{X}^{(2)}$  and  $(y_1, y_2) \in \mathbb{Y}^{(2)}$  linked via the transformation, as both original variables are strictly positive, we can extract the inequalities

$$0 < y_1 < y_2 < \infty.$$

4. The Jacobian for points  $(y_1, y_2) \in \mathbb{Y}^{(2)}$  is found from

$$D_y = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow |J(y_1, y_2)| = |\det D_y| = |1| = 1.$$

Note, here,  $J(x_1, x_2) = |\det D_x| = 1$  also so that again

$$|J(y_1, y_2)| = \frac{1}{|J(x_1, x_2)|}.$$

5. Finally, we have for  $0 < y_1 < y_2 < \infty$ ,

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= f_{X_1, X_2}(y_1, y_2 - y_1) \times 1 = f_{X_1}(y_1) \times f_{X_2}(y_2 - y_1) \quad \text{by independence} \\ &= \sqrt{\frac{1}{2\pi y_1}} \exp\left\{-\frac{y_1}{2}\right\} \sqrt{\frac{1}{2\pi(y_2 - y_1)}} \exp\left\{-\frac{(y_2 - y_1)}{2}\right\} \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{y_1(y_2 - y_1)}} \exp\left\{-\frac{y_2}{2}\right\}, \end{aligned}$$

and zero otherwise.

Here, for  $y_2 > 0$ ,

$$\begin{aligned}
 f_{Y_2}(y_2) &= \int f_{Y_1, Y_2}(y_1, y_2) dy_1 = \int_0^{y_2} \frac{1}{2\pi} \frac{1}{\sqrt{y_1(y_2 - y_1)}} \exp\left\{-\frac{y_2}{2}\right\} dy_1 \\
 &= \frac{1}{2\pi} \exp\left\{-\frac{y_2}{2}\right\} \int_0^{y_2} \frac{1}{\sqrt{y_1(y_2 - y_1)}} dy_1 \\
 &= \frac{1}{2\pi} \exp\left\{-\frac{y_2}{2}\right\} \int_0^1 \frac{1}{\sqrt{ty_2(y_2 - ty_2)}} y_2 dt \quad \text{setting } y_1 = ty_2 \\
 &= \frac{1}{2\pi} \exp\left\{-\frac{y_2}{2}\right\} \int_0^1 \frac{1}{\sqrt{t(1-t)}} dt \\
 &= \frac{1}{2} \exp\left\{-\frac{y_2}{2}\right\},
 \end{aligned}$$

as (by MAPLE, or further transformations)

$$\int_0^1 \frac{1}{\sqrt{t(1-t)}} dt = \pi.$$