M2S1 - ASSESSED COURSEWORK 1, 2011-12

The submission deadline is Monday 14th November, 2.00pm.
Please hand in to the Undergraduate Mathematics Student Office

1. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with common probability mass function

$$f_X(x) = P(X = x) = \frac{(1-p)^x}{x \log(1/p)},\ x = 1, 2, \ldots,$$

where $0 < p < 1$. Let N have the Poisson distribution with parameter μ, independently of $X_1, X_2, \ldots$.

Determine the moment generating function of $Y = \sum_{i=1}^{N} X_i$, and hence identify its distribution.

[*Hint.* Use the law of iterated expectation: $E(W) = E(E(W|U))$. You may assume that $\mu / \log p$ is an integer.] *[6 MARKS]*

2. Let X and Y be independent, identically distributed $N(\mu, \sigma^2)$ random variables.

Let $M = \frac{1}{2}(X + Y), V = (X - M)^2 + (Y - M)^2$.

By considering the joint moment generating function of $(M, X - M, Y - M)$, show that M and V are independent. *[3 MARKS]*

3. Let $X_1, \ldots, X_n$ be independent, identically distributed $N(\mu, 1)$.

Let

$$\bar{X} = n^{-1} \sum_{i=1}^{n} X_i,\ S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Determine the distributions of: (i) $n(\bar{X} - \mu)^2$; (ii) $\sum_{i=1}^{n} (X_i - \mu)^2$.

Show that

$$\sum_{i=1}^{n} (X_i - \mu)^2 = n(\bar{X} - \mu)^2 + \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Using a generalization of the result in Question 2 [which you should formulate carefully, but need not prove], and moment generating functions, determine the distribution of $\sum_{i=1}^{n} (X_i - \bar{X})^2$.

[6 MARKS]

4. Let A, B and C be independent random variables, uniformly distributed on $(0, 1)$. What is the probability that $Ax^2 + Bx + C$ has real roots?

[*Hint.* If X is uniformly distributed on $(0, 1)$, $-\log X$ is exponentially distributed. The sum of two independent exponentials is gamma.] *[5 MARKS]*