M2S1 - EXERCISES 8: SOLUTIONS

1. As $X_1, ..., X_n \sim Poisson(\lambda)$, and given that $T_1 = \bar{X}$, then using elementary properties of expectations, we have

$$E_{f_{T_1}}[T_1] = \frac{1}{n}\sum_{i=1}^{n} E_{f_{X_i}}[X_i] = \frac{1}{n}\sum_{i=1}^{n}\lambda = \lambda,$$

so that T_1 is an *unbiased* estimator of λ. Furthermore

$$\begin{aligned} T_2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\overline{X}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{(n-1)}\sum_{i=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 \\ &= \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n^2(n-1)}\sum_{i=1}^{n}\left(\sum_{i=1}^{n} X_i\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} X_i\right)^2. \end{aligned}$$

From properties of expectations, variances, and the Poisson distribution,

$$E_{f_X}\left[X^2\right] = Var_{f_X}[X] + \{E_{f_X}[X]\}^2 = \lambda + \lambda^2 = \lambda(\lambda+1).$$

Now, from properties of independent Poisson random variables $Y_n = \sum_{i=1}^{n} X_i \sim Poisson(n\lambda)$ so therefore, taking expectations in the above,

$$\begin{aligned} E_{f_{T_2}}[T_2] &= \frac{1}{n-1}\sum_{i=1}^{n} E_{f_{X_i}}\left[X_i^2\right] - \frac{1}{n(n-1)}E_{f_{Y_n}}\left[Y_n^2\right] = \frac{1}{n-1}\sum_{i=1}^{n}\lambda(\lambda+1) - \frac{1}{n(n-1)}n\lambda(n\lambda+1) \\ &= \frac{n}{n-1}\lambda(\lambda+1) - \frac{1}{n-1}\lambda(n\lambda+1) = \frac{1}{n-1}\left[n\lambda(\lambda+1) - \lambda(n\lambda+1)\right] = \lambda. \end{aligned}$$

2. From the theorem in lectures, and by properties of the Gamma distribution, we can write

$$V_n = \frac{(n-1)s_n^2}{\sigma^2} \sim \chi^2_{n-1} \equiv Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right) \Longrightarrow V_n = \sum_{i=1}^{n-1} X_i,$$

where $X_i \sim \chi^2_1 \equiv Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$, so that, using the Gamma expectation and variance results,

$$\mathrm{E}_{f_{X_i}}[X_i] = \frac{1/2}{1/2} = 1 = \mu, \qquad \mathrm{Var}_{f_{X_i}}[X_i] = \frac{1/2}{(1/2)^2} = 2 = \sigma^2,$$

say. Hence, by the Central Limit Theorem,

$$\frac{V_n - (n-1)\mu}{\sqrt{(n-1)\sigma^2}} = \frac{V_n - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} Z \sim N(0,1).$$

Hence, substituting in the definition for V_n,

$$\frac{\dfrac{(n-1)s_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} = \frac{\sqrt{n-1}\left(s_n^2 - \sigma^2\right)}{\sigma^2\sqrt{2}} \xrightarrow{d} Z \sim N(0,1),$$

and finally, by a location/scale transformation to $Z_n = \sigma^2 + \dfrac{\sigma^2\sqrt{2}}{\sqrt{n-1}}Z$, we have the result that approximately

$$s_n^2 \sim Z_n \sim N\left(\sigma^2, \frac{2\sigma^4}{n-1}\right).$$

3. $X_1, ..., X_n \sim Gamma(\alpha, \beta)$ so that $\mathrm{E}_{f_{X_i}}[X_i] = \alpha/\beta$ and $\mathrm{Var}_{f_{X_i}}[X_i] = \alpha/\beta^2$ and therefore

$$E_{f_{X_i}}\left[X_i^2\right] = \mathrm{Var}_{f_{X_i}}[X_i] + \left\{\mathrm{E}_{f_{X_i}}[X_i]\right\}^2 = \frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha(\alpha+1)}{\beta^2}.$$

Hence for the method of moments estimators $\hat{\alpha}_{MM}$ and $\hat{\beta}_{MM}$, need to simultaneously solve the following:

FIRST MOMENT $$\frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x} = \frac{\alpha}{\beta}$$

SECOND MOMENT $$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = (\overline{x})^2 + S^2 = \frac{\alpha(\alpha+1)}{\beta^2},$$

where $S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\overline{x})^2$.

Elementary algebra gives

$$\hat{\alpha}_{MM} = \frac{(\overline{x})^2}{S^2}, \qquad \hat{\beta}_{MM} = \frac{\overline{x}}{S^2}.$$

4. (i) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

STEP 2 $$\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0 \quad \Longrightarrow \quad \hat{\theta}_{ML} = -n/\sum_{i=1}^{n}\log x_i$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^{n}\log x_i}, \qquad \text{ESTIMATOR: } -\frac{n}{\sum_{i=1}^{n}\log X_i}.$$

(ii) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} (\theta+1)\, x_i^{-(\theta+2)} = (\theta+1)^n \left(\prod_{i=1}^{n} x_i\right)^{-(\theta+2)}$$

STEP 2 $$\log L(\theta) = n\log(\theta+1) - (\theta+2)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{n}{\theta+1} - \sum_{i=1}^{n}\log x_i = 0 \quad \Longrightarrow \quad \hat{\theta}_{ML} = n/\sum_{i=1}^{n}\log x_i - 1$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{n}{(\theta+1)^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n}\log x_i} - 1, \qquad \text{ESTIMATOR: } = \frac{n}{\sum_{i=1}^{n}\log X_i} - 1.$$

(iii) For $\theta > 0$

STEP 1 $$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta^2 x_i \exp\{-\theta x_i\} = \theta^{2n}\left(\prod_{i=1}^{n} x_i\right)\exp\left\{-\theta\sum_{i=1}^{n} x_i\right\}$$

STEP 2 $$\log L(\theta) = 2n\log\theta + \sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log L(\theta)\} = \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0 \qquad \Longrightarrow \qquad \hat{\theta}_{ML} = \frac{2n}{\sum\limits_{i=1}^{n} x_i}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log L(\theta)\} = -\frac{2n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{2n}{\sum\limits_{i=1}^{n} x_i}, \qquad \text{ESTIMATOR: } = \frac{2n}{\sum\limits_{i=1}^{n} X_i}.$$

(iv) Because of the constraint in the pdf that $x \leq \theta$

STEP 1 $$L(\theta) \quad \prod_{i=1}^{n} f_X(x_i;\theta) == \begin{cases} \prod\limits_{i=1}^{n} 2\theta^2 x_i^{-3} = 2^n\theta^{2n}\left(\prod\limits_{i=1}^{n} x_i^{-3}\right), & \theta \leq x_1,...,x_n, \\ 0, & \text{otherwise.} \end{cases}$$

STEP 2 $$\log L(\theta) = n\log 2 + 2n\log\theta - 3\sum_{i=1}^{n}\log x_i$$

At this point we note that the likelihood is monotonically increasing in θ, and hence the likelihood is maximized when θ is as **large** as possible but so that the constraint $\theta \leq x_1,...,x_n$ is still satisfied, hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \min\{x_1,...,x_n\}, \qquad \text{ESTIMATOR: } \min\{X_1,...,X_n\}.$$

(v) Noting the constraint in the pdf that $x \geq \theta_2$, we have

STEP 1 $$L(\theta_1,\theta_2) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta_1\theta_2^{\theta_1} x_i^{-(\theta_1+1)} = \theta_1^n\theta_2^{n\theta_1}\left(\prod_{i=1}^{n} x_i\right)^{-(\theta_1+1)}, \qquad \theta_2 \leq x_1,...,x_n$$

STEP 2 $$\log L(\theta_1,\theta_2) = n\log\theta_1 + n\theta_1\log\theta_2 - (\theta_1+1)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{\partial}{\partial\theta_1}\{\log L(\theta_1,\theta_2)\} = \frac{n}{\theta_1} + n\log\theta_2 - \sum_{i=1}^{n}\log x_i = 0$$

$$\Longrightarrow \quad \hat{\theta}_{1ML} = \frac{n}{\sum\limits_{i=1}^{n}\log x_i - n\log\hat{\theta}_{2ML}}$$

$$\frac{\partial}{\partial\theta_2}\{\log L(\theta_1,\theta_2)\} = \frac{n\theta_1}{\theta_2}$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when θ_2 is as **large** as possible, that is, when $\hat{\theta}_{2ML} = \min\{x_1, ..., x_n\}$. Hence

ESTIMATES:

$$\hat{\theta}_{1ML} = \frac{n}{\left[\sum\limits_{i=1}^{n} \log x_i - n \log\{\min\{x_1, ..., x_n\}\}\right]}$$

$$\hat{\theta}_{2ML} = \min\{x_1, ..., x_n\}$$

ESTIMATORS:

$$\theta_1 \quad \frac{n}{\left[\sum\limits_{i=1}^{n} \log x_i - n \log\{\min\{X_1, ..., X_n\}\}\right]}$$

$$\theta_2 \quad \min\{X_1, ..., X_n\}$$

5. Follow the four step procedure that can be summarized as follows: for an observed random sample $x_1, ..., x_n$ from a distribution represented by mass/density function $f_X(x;\theta)$

STEP 1 : Form the likelihood function $L(\theta)$

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta)$$

STEP 2 Take (natural) log to obtain $\log L(\theta)$

STEP 3: Find the value of θ at which $\log L(\theta)$ (and hence $L(\theta)$) is maximized within the parameter space Θ by differentiation

STEP 4: Check the **maximum** value has been found.

Formally, we define the **maximum likelihood estimate** of θ, $\hat{\theta}_{ML}$, as

$$\hat{\theta}_{ML} = \arg\max_{\theta} L(\theta).$$

Hence, for the $Poisson(\lambda)$ case

STEP 1 $$L(\lambda) = \prod_{i=1}^{n} f_X(x_i;\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum\limits_{i=1}^{n} x_i}}{\left(\prod\limits_{i=1}^{n} x_i!\right)}$$

STEP 2 $$\log L(\lambda) = -\sum_{i=1}^{n} \log x_i! - n\lambda + \left(\sum_{i=1}^{n} x_i\right)\log\lambda$$

STEP 3 $$\frac{d}{d\lambda}\{\log L(\lambda)\} = -n + \left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda} = 0 \qquad \Longrightarrow \qquad \hat{\lambda}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

STEP 4 $$\frac{d^2}{d\lambda^2}\{\log L(\lambda)\} = -\left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda^2} < 0 \qquad \text{for all } \lambda$$

Therefore

$$\text{ESTIMATE : } \hat{\lambda}_{ML} = \overline{x}, \qquad \text{ESTIMATOR: } \overline{X},$$

and from question 1 on this sheet, we know that if $T_1 = \overline{X} = \dfrac{1}{n}\sum\limits_{i=1}^{n} X_i$ then $E_{f_{T_1}}[T_1] = \lambda$ and T_1 is unbiased. Now, if

$$\tau = \tau(\lambda) = e^{-\lambda} \qquad \text{so that} \qquad \lambda = -\log\tau,$$

we can reformulate the likelihood in terms of τ, giving

$$\log L(\tau) = -\sum_{i=1}^{n} \log x_i! + n \log \tau + \left(\sum_{i=1}^{n} x_i \right) \log(-\log \tau)$$

and

$$\frac{d}{d\lambda}\left\{\log L(\tau)\right\} = \frac{\left(\sum\limits_{i=1}^{n} x_i\right)}{-\tau \log \tau} + \frac{n}{\tau} = 0 \Longrightarrow \qquad \hat{\tau}_{ML} = \exp\left\{ -\frac{1}{n}\sum_{i=1}^{n} x_i \right\} = \exp\left\{-\overline{x}\right\},$$

which can be shown to be the value that maximizes the likelihood, so that

$$\hat{\tau}_{ML}\left(\lambda\right) = \tau\left(\hat{\lambda}_{ML}\right).$$

6. (i) $X_1, ..., X_n \sim \textit{Exponential}(1/\theta)$ so that $\mathrm{E}_{f_{X_i}}\left[X_i\right] = \theta$ and hence, using standard mgf techniques, we have

$$X = \sum_{i=1}^{n} X_i \sim \textit{Gamma}\left(n, \frac{1}{\theta}\right) \qquad \Longrightarrow \qquad E_{f_X}[X] = \frac{n}{\frac{1}{\theta}} = n\theta,$$

so that if $T_1 = \overline{X} = \dfrac{1}{n}X$ then

$$E_{f_{T_1}}\left[T_1\right] = \frac{1}{n} n\theta = \theta,$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\left\{X_1, ..., X_n\right\}$, then previous order statistics results give that

$$F_{Y_1}(y) = 1 - \left\{1 - F_X(y)\right\}^n = 1 - \left\{1 - \left(1 - e^{-y/\theta}\right)\right\}^n = 1 - e^{-ny/\theta}, \qquad y > 0,$$

so that $Y_1 \sim \textit{Exponential}\left(\dfrac{n}{\theta}\right)$. Hence if $T_2 = nY_1$ then

$$E_{f_{Y_1}}\left[Y_1\right] = \frac{\theta}{n}, \qquad E_{f_{T_2}}\left[T_2\right] = n\frac{\theta}{n} = \theta,$$

and hence T_2 is an unbiased estimator of θ.

(ii) Straightforward calculations show that the MLE is given by $\hat{\theta} = n/T$, where T is given by $T = \sum_{i=1}^{n} X_i$ and is distributed as $Ga(n, \theta)$ [use moment generating functions if necessary]. We have that the pdf of T is

$$f_T(t;\theta) = \theta^n t^{n-1} e^{-\theta t}/\Gamma(n).$$

Note that since n is an integer $\Gamma(n) = (n-1)\Gamma(n-1)$. We have

$$E(1/T) = \int_0^\infty \frac{\theta^n t^{n-2} e^{-\theta t}}{\Gamma(n)} dt = \frac{\theta}{(n-1)} \int_0^\infty \frac{\theta^{n-1} t^{n-2} e^{-\theta t}}{\Gamma(n-1)} dt = \frac{\theta}{(n-1)},$$

since the integrand is the pdf of $Ga(n-1, \theta)$. Hence $\hat{\theta}$ has expectation $n\theta/(n-1) \neq \theta$, so the MLE is biased, but $(n-1)\hat{\theta}/n$ is unbiased.

Transformation gives that $2\theta \sum_{i=1}^{n} X_i$ is distributed as $Ga(n, 1/2)$ i.e. χ^2_{2n}, so it is pivotal. Let c_1 and c_2 be the $\alpha/2$ and $1-\alpha/2$ quantiles of χ^2_{2n}. We then have

$$P(c_1 < 2\theta T < c_2) = 1 - \alpha.$$

'Pivot' to obtain

$$P(c_1/(2T) < \theta < c_2/(2T)) = 1 - \alpha,$$

so that $(c_1/(2T), c_2/(2T))$ is a $100(1-\alpha)\%$ confidence interval. To test H_0: accept H_0 iff θ_0 is in the confidence interval constructed from the data sample.

7. We have

$$f_X(x) = \frac{1}{2}, \qquad \theta - 1 \leq x \leq \theta + 1, \qquad F_X(x) = \frac{x-(\theta-1)}{2} = \frac{x-\theta+1}{2}, \qquad \theta - 1 \leq x \leq \theta + 1.$$

$\mathrm{E}_{f_{X_i}}[X_i] = \theta$ (by integration, or by noting that the pdf is constant and hence symmetric about θ) and hence, using standard expectation techniques, we have that if $T_1 = \overline{X}$

$$\mathrm{E}_{f_{T_1}}[T_1] = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}_{f_{X_i}}[X_i] = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{1}{n}n\theta = \theta,$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\{X_1, ..., X_n\}$ and $Y_n = \max\{X_1, ..., X_n\}$, then previous order statistics results give that

$$f_{Y_1}(y) = n f_X(y)\{1-F_X(y)\}^{n-1} = n\frac{1}{2}\left\{1 - \frac{y-(\theta-1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1+\theta-y}{2}\right\}^{n-1}, \qquad \theta - 1 \leq y \leq \theta + 1,$$

and

$$f_{Y_n}(y) = n f_X(y)\{F_X(y)\}^{n-1} = n\frac{1}{2}\left\{\frac{y-(\theta-1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1-\theta+y}{2}\right\}^{n-1}, \qquad \theta - 1 \leq y \leq \theta + 1.$$

For the expectations,

$$\begin{aligned}
\mathrm{E}_{f_{Y_1}}[Y_1] &= \int_{\theta-1}^{\theta+1} y\frac{n}{2}\left\{\frac{1+\theta-y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 ((1+\theta)-2t)\, t^{n-1}\, 2\, dt, \qquad \text{setting } t = (1+\theta-y)/2 y = (1+\theta)-2t \\
&= (1+\theta)\int_0^1 n t^{n-1}\, dt - 2n\int_0^1 t^n dt \\
&= (1+\theta) - \frac{2n}{n+1},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}_{f_{Y_n}}[Y_n] &= \int_{\theta-1}^{\theta+1} y\frac{n}{2}\left\{\frac{1-\theta+y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 (2t-(1-\theta))\, t^{n-1}\, 2\, dt, \qquad \text{setting } t = (1-\theta+y)/2 y = 2t-(1-\theta) \\
&= 2\int_0^1 n t^n\, dt - (1-\theta)\int_0^1 n t^{n-1} dt \\
&= \frac{2n}{n+1} - (1-\theta),
\end{aligned}$$

so that if $M = (Y_1 + Y_n)/2$ then by properties of expectations

$$\mathrm{E}_{f_M}[M] = \frac{1}{2}\mathrm{E}_{f_{Y_1}}[Y_1] + \frac{1}{2}\mathrm{E}_{f_{Y_n}}[Y_n] = \left[\frac{1}{2}(1+\theta) - \frac{n}{n+1}\right] + \left[\frac{n}{n+1} - \frac{1}{2}(1-\theta)\right] = \theta,$$

and hence M is an unbiased estimator for θ.

[Alternatively, note that, by symmetry, we have that $Y_n - \theta$ has the same distribution as $\theta - Y_1$. Hence, $E(Y_n - \theta) = E(\theta - Y_1)$. Rearrange, using properties of expectation, to get $E(Y_1 + Y_n) = 2\theta$].

8. The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I(x_i < \theta) \equiv \frac{1}{\theta^n} I(\max\{x_1, \ldots, x_n\} < \theta).$$

Considering this as a function of θ, it is maximised at $\max\{x_1, \ldots, x_n\}$. So, the MLE is $Y_n = \max\{X_1, \ldots, X_n\}$.

Directly, we have $P(\hat{\theta}/\theta \leq y) = y^n$. Then we can check that the given interval satisfies the necessary property:

$$P(\hat{\theta} \leq \theta \leq \hat{\theta}/\alpha^{1/n}) = P(\hat{\theta}/\theta \geq \alpha^{1/n}) = 1 - P(\hat{\theta}/\theta \leq \alpha^{1/n}) = 1 - \alpha.$$

9. The key here is that $X_1 - \theta_1$ and $X_2 - \theta_2$ are IID $N(0,1)$, so that $(X_1 - \theta_1)^2 + (X_2 - \theta_2)^2$ is distributed as χ^2_2. Then, directly, we calculate: $P\{(\theta_1, \theta_2) \in S\} = \{P(|N(0,1)| \leq 2.236)\}^2 = \{1 - 2[1 - \Phi(2.236)]\}^2 = \{1 - 2 + (1 + \sqrt{0.95})\}^2 = 0.95$, using the hint. Also, $P\{(\theta_1, \theta_2) \in C\} = P(\chi^2_2 \leq 5.991) = 0.95$, on looking up Tables of the chi-squared distribution. [Note that χ^2_2 is actually exponential, with mean 2, so we can calculate the distribution function etc. directly, without the need for tables].

The sensible criterion to discriminate between S and C is *area* (more generally, the volume of a confidence set): we want the random set to be small, while containing the true (unknown) parameter value with the specified probability, here 95%. Note that here S and C have *fixed* (non-random) areas. The area of S is $(2 \times 2.236)^2 = 19.999$, while the area of C is $\pi \times 5.991 = 18.821$, so C is preferable according to the criterion.

10. (i) Suppose $X_1, \ldots, X_n$ are IID $N(\mu, \sigma^2)$. When σ^2 is unknown, we construct the confidence interval using the pivotal quantity

$$\frac{(\bar{X} - \mu)}{S/\sqrt{n}} \sim t_{n-1},$$

Student's $t-$distribution, on $n-1$ degrees of freedom. Then a $100(1-\alpha)\%$ confidence interval for μ is

$$(\bar{X} - t_{\alpha/2} S/\sqrt{n}, \bar{X} + t_{\alpha/2} S/\sqrt{n}),$$

in terms of the $1 - \alpha/2$ quantile of t_{n-1}. Note that $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n-1} \{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\}$.

Here, $n = 5$, $\alpha = 0.05$, so $t_{\alpha/2} = 2.776$. Also, $s = 1.2729$. Hence the width the the confidence interval is $\frac{2 \times 2.776 \times s}{\sqrt{5}} \approx 3.16$.

(ii) From lectures, the width of the confidence interval when σ^2 is known is $\frac{2 \times z_{\alpha/2} \times \sigma}{\sqrt{5}} = \frac{2 \times 1.96}{\sqrt{5}} \approx 1.75$, since $\sigma = 1$.

(iii) The probability in question is $P(S > z_{\alpha/2}\sigma/t_{\alpha/2}) = P(S > 1.96/2.776)$. But we know that $(n-1)S^2 \sim \chi^2_4$ (since $\sigma = 1$), so the required probability is $P(\chi^2_4 > 4(1.96/2.776)^2) = P(\chi^2_4 > 1.994) \approx 0.737$.