The Hull-White Swaption Formula

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19 March 2007

1 Exercise value and zero-vol valuation

The exercise value of the payer's swaption exercised at T_0 with payment dates $T_1 \dots T_n$ is

$$\left[1 - K\sum_{i=1}^{n}\theta_{i-1}p(T_0,T_i) - p(T_0,T_n)\right]^+ = A(T_0)\left[\frac{1-p(T_0,T_n)}{A(T_0)} - K\right]^+, \tag{1}$$

where A is the annuity

$$A(t) = \sum_{i=1}^{n}\theta_{i-1}p(t,T_i).$$

In the zero-volatility case we have

$$p(T_0,T_i) = \frac{p(0,T_i)}{p(0,T_0)} := \frac{D_i}{D_0}.$$

Hence

$$\left[\frac{1-p(T_0,T_n)}{A(T_0)} - K\right]^+ = \left[\frac{1-\frac{D_n}{D_0}}{\sum_i \theta_{i-1}\frac{D_i}{D_0}} - K\right]^+ = \left[\frac{D_0 - D_n}{\sum_i \theta_{i-1}D_i} - K\right]^+$$

and

$$A(T_0) = \sum_i \theta_{i-1}\frac{D_i}{D_0}.$$

The value at time 0 is

$$D_0 \times A(T_0) \times [\cdots]^+ = A(0)[S^f_{0,T_0} - K]^+$$

where $A(0) = \sum_i \theta_{i-1}D_i$ and S^f_{0,T_0} is the forward swap rate.

2 HW Zero-coupon Bond Option Volatility

Recalling that $B(t,\lambda) = \frac{1}{\lambda}(1-e^{-\lambda t})$, this is given by

$$\begin{aligned}
\overline{\sigma}^2 T_0 &= \int_0^{T_0}(B(T_1-t,\lambda) - B(T_0-t,\lambda))^2 dt \\
&= \int_0^{T_0}\frac{1}{\lambda^2}e^{2\lambda t}\left(e^{-\lambda T_0} - e^{-\lambda T_1}\right)^2 dt \\
&= \frac{1}{2\lambda^3}\left(e^{2\lambda T_0} - 1\right)\left(e^{-\lambda T_0} - e^{-\lambda T_1}\right)^2 \\
&= \frac{1}{2\lambda^3}\left(1 - e^{-2\lambda T_0}\right)\left(1 - e^{-\lambda(T_1 - T_0)}\right)^2
\end{aligned}$$

so that

$$\overline{\sigma} = B(T_1 - T_0,\lambda)\sqrt{\frac{B(T_0,2\lambda)}{T_0}}. \tag{2}$$

3 The Jamshidian Decomposition

The ZC bond $p(T_0, T_i)$ is expressed as $p(T_0, T_i) = p_i(r)$ where r is the short rate at time T_0. Let r^* be the unique value of r such that

$$K \sum_{i=1}^{n} \theta_{i-1} p_i(r^*) - p_n(r^*) = 1$$

and denote

$$\begin{aligned} \alpha_i &= \theta_{i-1} p_i(r^*), \quad i < n \\ \alpha_n &= (1 + \theta_{n-1}) p_i(r^*). \end{aligned}$$

Then the exercise value (1) is

$$\begin{aligned} \sum_{i=1}^{n} [\alpha_i - K\theta_{i-1} p_i(r)]^+ \quad &+ \quad [\alpha_n - (1 + K\theta_{n-1}) p_n(r)]^+ \\ &= \sum_{i=1}^{n} K\theta_{i-1} \left[\frac{\alpha_i}{K\theta_{i-1}} - p_i(r) \right]^+ + (1 + K\theta_{n-1}) \left[\frac{\alpha_n}{(1 + K\theta_{n-1})} - p_n(r) \right]^+, \end{aligned}$$

expressing the swaption as a linear combination of zero-coupon bond options.

Recall that in the HW model

$$p(t, T)(r) = \frac{p(0, T)}{p(0, t)} \exp \left(-B(t, T) \frac{\partial}{\partial t} p(0, t) - \frac{\sigma^2}{4\lambda^3} (e^{-\lambda T} - e^{-\lambda t})^2 (e^{2\lambda t} - 1) - B(t, T) r \right).$$

Using this formula, we can find the value of r^*, and hence the values of the α_i, by binary search, and then the ZC bond option values from the Black formula with volatility $\overline{\sigma}$ given by (2).