

# Option Pricing in Incomplete Markets

Mark H.A. Davis

## Abstract

In this chapter a general option pricing formula is proposed, using arguments based on marginal substitution value. By giving the investor an external objective in the form of a utility maximization problem we arrive at a unique price in situations where standard arbitrage arguments cannot be used. Further, we show using Markov process theory that the price can be expressed as a discounted expectation where both the measure and the discount rate are uniquely determined. Models with stochastic coefficients and transaction cost models are studied in detail.

## 1 A General Option Pricing Formula

The Black-Scholes option pricing formula depends on exact replication and is only applicable in complete markets. It expresses the option value as the expected discounted exercise value where the expectation is calculated using the uniquely defined "martingale measure". In incomplete markets, exact replication is impossible and holding an option is a genuinely risky business, meaning that no *preference independent* pricing formula is possible. In technical terms, the problem is that no *unique* martingale measure exists. A variety of approaches have been suggested to get round this problem, none of them perhaps entirely satisfactory. Here we show that if option pricing is imbedded in a utility maximization framework, i.e. the potential option purchaser's attitude to risk is specified, then a unique measure emerges in a very natural way.

An investor with concave utility function  $U$  and starting with initial cash endowment  $x$  forms a dynamic portfolio whose cash value at time  $t$  is  $X_t^\pi(t)$  when he uses trading strategy  $\pi \in T$ , where  $T$  denotes the set of admissible trading strategies. His objective is to maximize expected utility of wealth at a fixed final time  $T$ ; we denote

$$V(x) = \sup_{\pi \in T} E[U(X_T^\pi(T))]. \quad (1)$$

Throughout the paper it will be assumed that the utility function  $U$  is non-decreasing and  $C^2$  on  $R_+$  with  $U' > 0, \lim_{x \rightarrow 0} U'(x) = \infty$  and

$\lim_{x \rightarrow \infty} U'(x) = 0$ . We ask the question whether the maximum utility in (1) can be increased by the purchase (or short-selling) of a European option whose cash value at time  $T$  is some non-negative random variable  $B$ , the purchase price at time zero being  $p$ . We use a "marginal rate of substitution" argument:  $p$  is a fair price for the option if diverting a little of his funds into it at time zero has a neutral effect on the investor's achievable utility. This is an entirely traditional approach to pricing in economics — see [6] for references — but does not appear to have been used much in an option pricing context. To state the definition in precise terms, we need the function  $W$  given as

$$W(\delta, x, p) = \sup_{\pi \in T} EU \left( X_{x-\delta}^\pi(T) + \frac{\delta}{p} B \right).$$

**Definition 1** Suppose that for each  $(x, p)$  the function  $\delta \mapsto W(\delta, x, p)$  is differentiable at  $\delta = 0$  and there is a unique solution  $\hat{p}(x)$  of the equation

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0.$$

Then  $\hat{p}(x)$  is the fair option price at time 0.

This definition will clearly reproduce the Black-Scholes value if perfect hedging is possible. The argument is as follows: suppose  $p_0$  is the perfect-replication value and the option is offered for  $p$ . The investor buys  $\delta/p$  options with cash  $\delta$ , investing the remaining  $x - \delta$  in a portfolio. A moment's thought shows that his optimal procedure is to short the hedging portfolio, whose value is  $\delta p_0/p$  and invest his cash fund of  $x + \delta(p_0/p - 1)$  optimally, attaining an expected utility of  $V(x + \delta(p_0/p - 1))$ . (The option and short hedging fund have equal and opposite value at time  $T$ .) The marginal rate of substitution is therefore

$$\frac{d}{d\delta} V \left( x + \delta \left( \frac{p_0}{p} - 1 \right) \right) \Big|_{\delta=0} = \left( \frac{p_0}{p} - 1 \right) V'(x).$$

Evidently, this is equal to zero exactly when  $p = p_0$ .

In general, if the investor diverts  $\delta$  into options and uses trading strategy  $\pi$  then his expected utility is

$$E \left[ U \left( X_{x-\delta}^\pi(T) + \frac{\delta}{p} B \right) \right] = E[U(X_{x-\delta}^\pi(T))] + \frac{\delta}{p} E[U'(X_{x-\delta}^\pi(T)B)] + o(\delta). \quad (2)$$

We now need the following lemma.

**Lemma 2** Let  $f : A \times R \rightarrow R$  be a function, where  $A$  is some set, and for  $\delta \in R$  define

$$v(\delta) := \sup_{\pi \in A} f(\pi, \delta).$$

Suppose that, for some  $\delta_0 \in R$ ,  $v$  is differentiable at  $\delta_0$ , there exists  $\pi^* \in A$  such that  $v(\delta_0) = f(\pi^*, \delta_0)$  and the function  $\delta \mapsto f(\pi^*, \delta)$  is differentiable at  $\delta_0$ . Then

$$\frac{d}{d\delta} v(\delta_0) = \frac{\partial}{\partial \delta} f(\pi^*, \delta_0).$$

We can now give a general option pricing formula based on Definition 1.

**Theorem 3** Suppose that  $V$  is differentiable at each  $x \in R_+$  and that  $V'(x) > 0$ . Then the fair price  $\hat{p}(x)$  of Definition 1 is given by

$$\hat{p} = \frac{E[U'(X_x^*(T))B]}{V'(x)}. \quad (3)$$

The proof is obtained by evaluating the derivative with respect to  $\delta$  of the maximum utility at  $\delta = 0$ , using (2) and Lemma 2, giving a value of

$$-V'(x) + \frac{1}{p} E_x[U'(X_T^* B)],$$

from which (3) follows.

## 2 The Standard Model

This model is the one described in, for example, [7, 9]. Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space on which is defined a  $d$ -dimensional Brownian motion  $(W_t)_{t \geq 0}$ , with natural filtration  $(\mathcal{F}_t)$ . The available instruments in the market consist of a bond whose price  $S_0(t)$  satisfies

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1, \quad (4)$$

and  $m$  stocks ( $m \leq d$ ) whose prices satisfy the SDEs

$$dS_i(t) = S_i(t)[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t)], \quad i = 1, \dots, m. \quad (5)$$

The processes  $r(t)$ ,  $b_i(t)$ ,  $\sigma_{ij}(t)$  are assumed to be bounded, measurable and  $\mathcal{F}_t$ -adapted, with the matrix  $\sigma = \sigma_{ij}$  having full rank and being such that the entries of the matrix  $(\sigma(t)\sigma^T(t))^{-1}$  are bounded. We now define the following processes ( $\mathbf{1}$  is the  $m$ -vector each of whose entries is equal to 1, and superscript ' $T$ ' denotes transpose):

$$\theta(t) = \sigma^T(t)(\sigma(t)\sigma^T(t))^{-1}(b(t) - r(t)\mathbf{1})$$

$$\beta(t) = \exp\left(-\int_0^t r(s)ds\right)$$

$$Z_0(t) = \exp\left(-\int_0^t \theta^T(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right)$$

$$W_0(t) = W(t) + \int_0^t \theta(s)ds.$$

The class  $\mathcal{T}$  of admissible trading strategies consists of all adapted  $m$ -vector processes  $\pi(t)$  satisfying

$$\int_0^t \|\pi(s)\|^2 ds < \infty.$$

The interpretation here is that  $\pi_i(t)$  represents the wealth invested in the  $i$ th stock at time  $t$ . The portfolio value process  $X^\pi(t)$  corresponding to a strategy  $\pi \in \mathcal{T}$  and with initial wealth  $x$  is the unique solution of the SDE

$$dX^\pi(t) = r(t)X^\pi(t)dt + \pi^T(t)\sigma(t)dW_0(t), \quad X^\pi(0) = x. \quad (6)$$

Let us first consider the complete market case  $m = d$ . The solution of the utility maximization problem (1) is then described as follows [7]. Let  $I$  denote the inverse function of the gradient of the utility function,  $I = (U')^{-1}$ . Now define

$$\mathcal{H}(y) = E[\beta(T)Z_0(T)I(y\beta(T)Z_0(T))]$$

and  $\mathcal{Y}(x) = \mathcal{H}^{-1}(x)$ . The terminal wealth corresponding to the optimal strategy  $\pi^*$  is  $X_{\pi^*}^*(T) = I(\mathcal{Y}(x)\beta(T)Z_0(T))$ , and hence the value function is

$$V(x) = E\left[U\left(I(\mathcal{Y}(x)\beta(T)Z_0(T))\right)\right].$$

A simple computation using the fact that  $U'(I(x)) = x$  shows that  $V'(x) = \mathcal{Y}(x)$  and the pricing formula (3) becomes

$$\hat{p} = E[\beta(T)Z_0(T)B],$$

which, as claimed earlier, is just the Black-Scholes price.

Now consider the incomplete case  $m < d$ , which is analyzed by Karatzas *et al.* in [9]. They show that under certain conditions there is a least favourable completion of the market, i.e. a specification of  $d - m$  fictitious stocks in such a way that the utility-maximizing investor would place zero investment in these stocks even if they were available. The maximizing strategy can then be expressed by formulas similar to the above in terms of the completed market, and the following result is readily shown; see Rabeau [10] for details.

**Theorem 4** For the standard model with  $m < d$ , the option price (3) coincides with the Black-Scholes price computed for the least favourably completed market.

### 3 Multiplicative Functionals

#### 3.1 The Markov Case

Returning to general formula market models, let us suppose that the optimally controlled portfolio process  $X_t^*(t)$  is one component – say the first, denoted  $x_t^1$  – of some Markov process  $x_t$  in  $R^n$ . The pricing formula (3) then has a simple interpretation as a multiplicative functional (MF) transformation of the Markov semigroup. A discussion of MFs adequate for this application can be found in [2]; the standard reference is Blumenthal and Gettoor [1]. The treatment here is at an informal level, making various assumptions that would need checking in specific applications.

Let  $\mathcal{A}_t$  denote the extended generator of  $(x_t)$  and  $\hat{\mathcal{A}}$  be the generator of the corresponding time-space process  $(t, x_t)$ , i.e.

$$\hat{\mathcal{A}} = \frac{\partial}{\partial t} + \mathcal{A}_t.$$

Let

$$V(t, x) = E_{t,x}[U(x_T^1)],$$

and suppose that  $V$  is differentiable in  $x$  and that  $(\partial/\partial x^1)V(t, x) > 0$ . Defining

$$h(t, x) = \log \frac{\partial}{\partial x^1} V(t, x),$$

we see that the pricing formula is expressed as

$$\hat{p}(x) = e^{-h(0,x)} E_{0,x}[e^{h(T,x^1)} B]. \quad (7)$$

If  $P_{s,t}$  denotes the Markov semigroup of  $(x_t)$  then (7) is just integration of  $B$  with respect to the measure defined by a new semigroup

$$P_{s,t}^h f(x) = e^{-h(s,x)} P_{s,t}[e^h f](x)$$

obtained by transformation of  $P_{s,t}$  by the MF  $m_{st} = \exp(h(t, x_t) - h(s, x_s))$ . We can obtain more insight into this transformation by factoring this MF. Recall that if  $f$  is a function in the domain  $\mathcal{D}(\hat{\mathcal{A}})$  of the extended generator  $\hat{\mathcal{A}}$  then the process  $(f(t, x_t))$  is expressed as

$$f(t, x_t) = f(0, x) + \int_0^t \hat{\mathcal{A}}(s, x_s) ds + C_t^f, \quad (8)$$

where  $C_t^f$  is a local martingale. For simplicity, let us assume that  $(C_t^f)$  is a continuous local martingale for all  $f \in \mathcal{D}(\hat{\mathcal{A}})$ . We then have the following result.

**Theorem 5** Suppose that  $h, h^2 \in \mathcal{D}(\hat{\mathcal{A}})$ . Then the price  $\hat{p}$  of (3) is expressed as

$$\hat{p} = \bar{E} \left[ \exp \left( - \int_0^t \gamma^h(s, x_s) ds \right) B \right]$$

where

$$\gamma^h = (h-1)\hat{\mathcal{A}}h - \frac{1}{2}\hat{\mathcal{A}}h^2 \quad (9)$$

and  $\bar{E}$  denotes expectation with respect to the measure  $\bar{P}$  defined by

$$\frac{d\bar{P}}{dP} = \exp \left( C_T^h - \frac{1}{2} \langle C^h, C^h \rangle_T \right).$$

PROOF: Writing  $h = h(t, x_t)$  etc., we have as in (8)

$$dh^2 = \hat{\mathcal{A}}h^2 dt + dC_t^{h^2},$$

whereas applying the Ito formula to the product  $h \cdot h$  we obtain

$$d(h \cdot h) = 2h(\hat{\mathcal{A}}h dt + dC_t^h) + d\langle C^h, C^h \rangle_t.$$

Identifying the bounded variation and local martingale terms in these two decompositions shows that

$$\langle C^h, C^h \rangle_t = \int_0^t (\hat{\mathcal{A}}h^2 - 2h\hat{\mathcal{A}}h) ds. \quad (10)$$

This is in fact a standard result, the operator in the integrand being known as the *opérateur carré du champ*. Using (10) we see that

$$h(t, x_t) - h(0, x) = C_t^h - \frac{1}{2} \langle C^h, C^h \rangle_t - \int_0^t \gamma^h(s, x_s) ds$$

where  $\gamma^h$  is given by (9). The result follows.

Formula (5) expresses the price as a discounted expectation where both the measure and the discount rate are uniquely specified by the utility maximization problem.

#### 3.2 Example: The Standard Model with Random Coefficients

As is well known, the utility maximization problem for the standard model of Section 2 is explicitly solvable for logarithmic utility  $U(x) = \log x$ . Let us consider a single risky asset, so that  $m = 1$ . The bond, stock and wealth equations are given by (4,5,6); define  $b = b_1$ ,  $\sigma = \sigma_{11}$  and  $u = \pi/X^\pi$  (the proportion of wealth invested in the risky asset) to simplify the notation. Then (6) becomes

$$dX^u = (r + (b-r)u)X^u dt + uX^u \sigma dw,$$

so that by the Ito formula

$$d \log X^u = \left( r + (b-r)u - \frac{1}{2}\sigma^2 u^2 \right) dt + u\sigma dw. \quad (11)$$

Define

$$\lambda(t) = \frac{b(t) - r(t)}{\sigma(t)}.$$

The 'dt' integrand in (11) is maximized pointwise by  $u = u^* := \lambda/\sigma$  and the value of the integrand is then  $(r + \lambda^2/2)$ . It follows easily that

$$V(t, x) = \log x + E \int_t^T \left( r(t) + \frac{1}{2}\lambda^2(t) \right) dt. \quad (12)$$

In Section 2 the coefficients  $r(t)$  etc. were allowed to be random in an essentially arbitrary way. To apply the results of this section we need to assume that they are deterministic functions of some underlying Markovian 'state variable'. (From a modelling point of view this is hardly a restriction since computation would rarely be feasible in any other case.) Thus, assume there is a Markov process  $(y_t)$  with extended generator  $\mathcal{G}$  such that  $r(t) = r(y_t)$ ,  $\sigma(t) = \sigma(y_t)$  and  $\lambda(t) = \lambda(y_t)$ , where  $r(\cdot)$ ,  $\sigma(\cdot)$ ,  $\lambda(\cdot)$  are bounded measurable functions. Further, assume that  $(y_t)$  and  $(w_t)$  are independent processes (this is a significant restriction - see below). Then the generator of the joint space-time process  $(t, z_t) := (t, X_t^{u^*}, y_t)$  is

$$\hat{A}f(t, x, y) = \frac{\partial f}{\partial t} + (r(y) + \lambda^2(y))x \frac{\partial f}{\partial x} + \frac{1}{2}\lambda^2(y)x^2 \frac{\partial^2 f}{\partial x^2} + \mathcal{G}f(x, y).$$

To allow for dependence on the initial state of the  $y_t$  process, the utility maximization value function  $V$  of (12) should be written as

$$V(t, x, y) = \log x + E_{t,y} \int_t^T \left( r(t) + \frac{1}{2}\lambda^2(t) \right) dt. \quad (13)$$

Thus

$$\begin{aligned} h(t, x, y) &= \log \frac{\partial}{\partial x} V(t, x, y) \\ &= -\log x, \end{aligned}$$

so that

$$\hat{A}h = -(r + \frac{1}{2}\lambda^2) \quad (14)$$

and

$$\hat{A}h^2 = (2r + \lambda^2) \log x + \lambda^2. \quad (15)$$

From (9), (14) and (15) we find that

$$\gamma^h(t) = r(y_t),$$

while from the Ito formula, or by applying (10),

$$\frac{d\tilde{P}}{dP} = \exp \left( C_T^h - \frac{1}{2} \langle C^h, C^h \rangle_T \right) = \exp \left( - \int_0^T \lambda dw_s - \frac{1}{2} \int_0^T \lambda^2 ds \right).$$

Under measure  $\tilde{P}$  the stock price process satisfies

$$dS_t = r(y_t)S_t dt + \sigma(y_t)S_t d\tilde{w}_t \quad (16)$$

where  $\tilde{w}_t$  is a  $\tilde{P}$ -Brownian motion, and Theorem 5 gives the option price as

$$\tilde{p}(x) = \tilde{E}_{0,x,y} \left[ e^{-\int_0^T r(y_s) ds} B \right]. \quad (17)$$

This is a striking result: the  $\tilde{P}$ -stock price model (16) is just the original model with the drift  $b$  replaced by the riskless rate  $r$ , and (17) expresses the price as the  $\tilde{P}$  expectation discounted at the riskless rate. This is the same expression as the Black-Scholes formula (valid when  $r$  etc are deterministic) even though the present model is not complete and there is no interpretation in terms of perfect hedging. Note, however, that the result depends on  $w_t$  and  $y_t$  being independent: if they are correlated then the second term in (13) also depends on  $x$  and the expressions given for  $\hat{A}h$  etc will be substantially different.

## 4 Transaction Cost Models

We can apply the pricing formula immediately to a model with transaction costs as considered in [3, 5, 4]. In this model the stock price  $S$ , number of shares held  $y$  and cash  $x$  satisfy the following equations:

$$\begin{aligned} dS &= bSdt + \sigma Sdw \\ dy &= dL - dM \end{aligned} \quad (18)$$

$$dx = (rx - c)dt - (1 + \lambda)dL - (1 - \mu)dM.$$

Here  $w$  is a standard Brownian motion,  $c_t$  is the consumption rate at time  $t$  and  $L_t, M_t$  are the cumulative purchases and sales respectively of stock units on the interval  $[0, t]$ . For standard application of the formula (3) we take  $c_t \equiv 0$ . However, we can also obtain a modified formula based on the utility maximization problem considered in [3], namely calculating

$$v(x, y, S) = \sup_{(c, L, M)} \frac{1}{\gamma} E \int_0^\infty e^{-\mu t} c_t^\gamma dt.$$

Thus we are maximizing the infinite-horizon discounted utility of consumption as measured by the utility function  $c^\gamma/\gamma$ , where  $\gamma < 1, \gamma \neq 0$ . A pricing

formula based on this utility maximization problem can be derived by the same approach as above, and it is

$$\hat{p}(x, y, S) = \frac{e^{-rT} v_x(x_T^*, y_T^*, S_T)}{v_x(x, y, S)}. \quad (19)$$

Here  $x_T^*, y_T^*$  is the composition of the optimal portfolio at time  $T$ , starting at  $x, y$  at time 0. By using the analysis of the utility maximization problem in [3], Rabeau obtains in [10] the following result.

**Theorem 6** For the transaction cost model (18), the option price  $\hat{p}$  of (19) is given by

$$\hat{p}(x, y, S) = \bar{E}[e^{-rT} B]$$

where

$$\frac{d\bar{P}}{dP} = \exp\left(\int_0^T \frac{1}{v_x} (\sigma y_t^* v_{xy}) dt - \frac{1}{2} \int_0^T \left(\frac{\sigma y_t^* v_{xy}}{v_x}\right)^2 dt\right)$$

As in the random coefficients model of Section 3.2, the discount rate specified by the formula is always the riskless rate  $r$ . It is not at all obvious from the general multiplicative functional decomposition of Theorem 5 why this should be so.

## 5 Concluding Remarks

### 5.1 Hedging

Having given a price, the trader will want to know how to hedge the portfolio. In incomplete markets this cannot be done exactly, but the utility function gives a measure of the riskiness of an imperfect hedge. If  $\epsilon$  options are written at price  $\hat{p}$  given by (3) then the best trading strategy from this point of view will be  $\pi^*$  maximizing

$$EU(X_{x+\epsilon\hat{p}}^* - \epsilon B).$$

As pointed out in Section 1, this strategy will amount to perfect replication of the option if this is possible.

For the transaction cost models discussed in Section 4 above, these utility maximization problems have been considered in [5, 4]. In particular, [4] gives a detailed account of discretization methods based on binomial approximation and subsequent solution of the utility maximization problems by dynamic programming.

The algorithms described in [4] are computationally intensive. A promising topic for future research is fast computation of nearly optimal strategies, perhaps by parametrising a class of 'good' strategies in some simple way and searching for best parameters.

## 5.2 Arbitrage

An obvious question is whether an option priced at the value  $\hat{p}$  of Definition 1 affords any arbitrage opportunities. This and many other matters are taken up in a recent paper by Karatzas and Kou [8], written since the first version of the present article. For the standard model with constraints on portfolio choice they show that for each option  $C$  there is an interval  $I_C = [p_0, p_1]$  such that any  $p \in I_C$  is a no-arbitrage price.  $I_C$  reduces to a single point in the case of a complete market. The price  $\hat{p}$  lies in  $I_C$  if a certain superposition principle holds: if  $\pi_1, \pi_2$  are admissible strategies with initial wealth  $x_1, x_2$  respectively, then there is an admissible strategy  $\pi$  starting from  $x_1 + x_2$  such that

$$X_{x_1}^{\pi_1}(T) + X_{x_2}^{\pi_2}(T) = X_{x_1+x_2}^{\pi}(T) \quad \text{a.s.}$$

This is a very mild condition, but it is possible to construct constraint sets for which  $\hat{p}$  fails to be arbitrage-free.

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