Chapter 7

7 Complex Numbers **80**

7.1 Complex Numbers: Basic definitions and properties 80

7.1.1 Polar form 82

7.1.2 Exponential form 83

7.1.3 Geometric properties 84

7.2 De Moivre's formula 85

7.2.1 Multiple angle formulas 85

7.3 Roots 86

7.4 Hyperbolic trigonometric functions 87

Chapter 7

Complex Numbers

One of the most elementary algebraic equations that does not have any solution is x^2+1. This is because the number $\sqrt{-1}$ does not exist. That is, there is no real number $x \in \mathbb{R}$ such that $x^2 = 1$. In order to tackle this problem, complex numbers were introduced. Indeed, complex numbers are an extension of real numbers where negative numbers have (two) square roots. Moreover (see Theorem 139), any algebraic equation $p(x) = 0$ given by a polynomial $p \in \mathbb{R}[x]$ of degree $n \in \mathbb{N}$ admits as many solutions as its degree, n.

7.1 Complex Numbers: Basic definitions and properties.

The goal of this section is introducing a set of numbers extending real numbers $\mathbb{R}$ where negative numbers will have (two) square roots as positive numbers do. Since we want to operate this new numbers in the same way as we manipulate real numbers, we will equip them with an addition and a multiplication satisfying the usual commutative, associative, and distributive properties. This implies that, in order to define the square root of a negative number,

$$\sqrt{-a}, \quad a > 0$$

it is enough to define the square root of -1,

$$\sqrt{-a} = \sqrt{(-1)\,a} = \sqrt{-1}\sqrt{a}, \quad a > 0.$$

Definition 125 *We define i as*

$$i := \sqrt{-1}$$

and we call it the ***imaginary unit****. That is, $i^2 = -1$. We define the set $\mathbb{C}$ of* ***complex numbers*** *as the set of expressions of the form*

$$z = a + ib, \quad a, b \in \mathbb{R}. \tag{7.1}$$

Given $z = a + ib$, a is called the ***real part*** *of z and is denoted by $\mathrm{Re}(z)$; b is called the* ***imaginary part*** *and is denoted by $\mathrm{Im}(z)$.*

Observe that we can think of real numbers $\mathbb{R}$ as the subset of complex numbers with imaginary part equal to 0,

$$\mathbb{R} = \{z \in \mathbb{C} : \mathrm{Im}\,(z) = 0\}\,.$$

Equation (7.1) is sometimes called the ***standard form*** of a complex number.

The next step is to define the addition and multiplication of complex numbers.

Definition 126 *Let $z, w \in \mathbb{C}$ be two complex numbers such that*

$$z = a + ib \quad \textit{and} \quad w = c + id$$

where $a, b, c, d \in \mathbb{R}$. We define

$$z + w := (a + c) + i\,(b + d) \quad \textit{(addition)}, \tag{7.2}$$

$$z \cdot w := (ac - bd) + i\,(bc + ad) \quad \textit{(multiplication)}. \tag{7.3}$$

That is, we define the sum $z + w$ of two complex numbers $z, w \in \mathbb{C}$ as the complex number whose real and imaginary part is the sum of the real and imaginary parts of z and w respectively. On the other hand, the multiplication is defined so that the distributive property holds and $i^2 = -1$. Indeed, applying the distributive property,

$$\begin{aligned} zw &= (a + ib)\,(c + id) = ac + iad + ibc + i^2 bd \\ &= ac + i\,(ad + bc) - bd = (ac - bd) + i\,(ad + bc) \end{aligned}$$

which coincides with (7.3).

Remark 127 *Observe that the product and the addition of two complex numbers can always be written in the standard form, which means that $\mathbb{C}$ is* ***closed*** *under addition and multiplication. In particular, the powers of $i \in \mathbb{C}$ must be written in the standard form as follows*

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = i^2 i^2 = 1, \ldots$$

That is

$$i^n = \begin{cases} 1 & \textit{if } n = 4k \\ i & \textit{if } n = 4k + 1 \\ -1 & \textit{if } n = 4k + 2 \\ -i & \textit{if } n = 4k + 3 \end{cases} \qquad k \in \mathbb{N}$$

In order to divide complex numbers we have to define the inverse $1/z$ of a complex number $z \in \mathbb{C}$, since dividing is just multiplying by the inverse. Note that, in principle,

$$\frac{1}{z} = \frac{1}{a + ib}, \quad z = a + ib, \tag{7.4}$$

is not formally a complex number because it is not written in the standard form. However, (7.4) can be easily rewritten in the standard from just multiplying both numerator and denominator by $a - ib$ such that

$$\frac{1}{z} = \frac{1}{(a + ib)}\frac{a - ib}{(a - ib)} = \frac{a - ib}{a^2 - i^2 b^2 + i(ab - ba)} = \frac{a - ib}{a^2 + b^2}.$$

Therefore,

$$\frac{1}{z} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

and

$$\text{Re}\left(\frac{1}{z}\right) = \frac{a}{a^2+b^2}, \quad \text{Im}\left(\frac{1}{z}\right) = -\frac{b}{a^2+b^2}.$$

Definition 128 *Given $z = a + ib \in \mathbb{C}$, the number $\overline{z} := a - ib \in \mathbb{C}$ is called the* ***conjugate*** *of z.*

Remark 129 *Observe that, in order to eliminate the imaginary part of the denominator in a complex number, we multiply both numerator and denominator by the conjugate of that denominator.*

Example 130 If $z = 2 + 3i$, we have

$$\frac{1}{2+3i} = \frac{1}{2+3i}\frac{2-3i}{2-3i} = \frac{2-3i}{2^2+3^2} = \frac{2}{13} - i\frac{3}{13}.$$

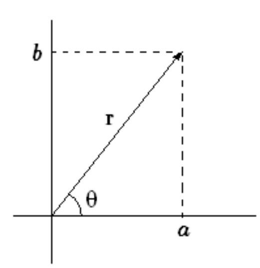
7.1.1 Polar form

Since complex numbers $z = a + ib \in \mathbb{C}$ can be represented through a pair of real numbers $a, b \in \mathbb{R}$, we usually describe complex numbers geometrically as points in the plane $\mathbb{R}^2$ where the number $z = a + ib$ corresponds to the point with x-coordinate a and y-coordinate b. Note that, according to the definition of complex numbers, each complex number $z \in \mathbb{C}$ corresponds to a unique point and vice versa, each point $(a, b) \in \mathbb{R}^2$ corresponds to a unique $z \in \mathbb{C}$. This representation of complex numbers allows us to write them in ***polar form***: If $z = a + ib$, then

$$z = r(\cos\theta + i\sin\theta)$$

where

$$r = \|z\| = \sqrt{a^2+b^2} \quad \text{and} \quad \tan\theta = \frac{b}{a}.$$



Polar representation of a complex number

The number r is called the ***modulus*** of z and θ the ***argument***. They are denoted by $\|z\|$ and $\arg(z)$ respectively. θ is the angle between the positive horizontal semi-axis and the line joining the origin to the point (a, b) measured anticlockwise in radians.

7.1.2 Exponential form

A third, very useful way of writing a complex number is a consequence of *Euler's formula* and it allows us to write a complex number using the exponential function. Recall that the exponential function e^x admits a power series expansion with infinite radius of convergence,

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}.$$

We can use the series expansion of e^x to define the exponential of any complex number.

Proposition 131 (Euler's formula) *For any* $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

As a consequence of Euler's formula, a complex number $z \in \mathbb{C}$ given in polar form $z = r(\cos\theta + i\sin\theta)$ can be written as

$$z = r(\cos\theta + i\sin\theta) = r\, e^{i\theta}\,. \tag{7.5}$$

Equation (7.5) is sometimes called the ***exponential form*** of a complex number.

Polar or exponential representations of complex numbers are more convenient to calculate the product or division of two complex numbers. Indeed, if

$$z = r(\cos\theta + i\sin\theta) = r\, e^{i\theta}$$
$$w = s(\cos\varphi + i\sin\varphi) = s\, e^{i\varphi}$$

are two complex numbers, then

$$zw = \left(r\, e^{i\theta}\right)\left(s\, e^{i\varphi}\right) = rs\, e^{i(\theta+\varphi)}$$

so that $\|zw\| = \|z\|\,\|w\|$ and $\arg(zw) = \arg(z) + \arg(w)$. On the other hand, it is now immediate to check that

$$\frac{1}{z} = \frac{1}{r}\, e^{-i\theta}$$

so

$$\frac{w}{z} = \frac{s}{r}\, e^{i(\varphi-\theta)}\,.$$

Example 132 Let $z = 1+i$ and $w = 2-3i$ be two complex numbers. We are going to compute zw, z/w, and z^{10}. In standard form, we have

$$zw = (1+i)(2-3i) = 2 - 3i + 2i - 3i^2 = 5 - i$$

and

$$\frac{z}{w} = \frac{1+i}{2-3i}\frac{2+3i}{2+3i} = \frac{2+3i+2i+3i^2}{2^2+3^2} = \frac{1+5i}{13} = \frac{1}{15} + i\frac{5}{13}.$$

On the other hand, to compute z^{10} is better to express it in the exponential form

$$z = \sqrt{1^2 + 1^2}\left(\cos\left(\tan^{-1}(1)\right) + i\sin(\tan^{-1}(1))\right) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

so

$$z = \sqrt{2}\, \mathrm{e}^{i\frac{\pi}{4}}\,.$$

We take powers applying the usual rules,

$$z^{10} = \left(\sqrt{2}\, \mathrm{e}^{i\frac{\pi}{4}}\right)^{10} = \sqrt{2}^{10}\, \mathrm{e}^{i10\frac{\pi}{4}} = 2^5\, \mathrm{e}^{i\frac{5\pi}{2}}\,.$$

However, observe that from Euler's formula

$$\mathrm{e}^{i\frac{5\pi}{2}} = \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

so, reverting to exponential notation, we have

$$z^{10} = 32i.$$

7.1.3 Geometric properties

The fact that complex numbers *live* on the plane instead of the line gives them a much richer geometry than that of the real numbers. It is very useful to have a geometric picture of the *complex plane* when solving certain equations involving complex numbers, even though the calculations may be completely analytical.

Example 133 Describe the regions of the complex plane where $\left\|z^2\right\| = 5\left\|z\right\|$.

Recall that $\|z\|$ is just the distance of the point z from the origin and therefore $\left\|z^2\right\| = \|z\|^2$. The equation $\left\|z^2\right\| = 5\|z\|$ then reads $\|z\|^2 = 5\|z\|$. $z = 0$ is clearly a solution and, for $z \neq 0$, we can simplify $\|z\|$ and we get $\|z\| = 5$, which is the the circle of radius 5.

Example 134 Describe the region of the complex plane where $\|z - i\| > \|z + i\|$.

Writing $z = x + iy$, we have $z - i = x + i(y - 1)$ and $z + i = x + i(y + 1)$. Moreover, $\|z\| = \sqrt{x^2 + y^2}$ so $\|z - i\| > \|z + i\|$ is equivalent to

$$\sqrt{x^2 + (y-1)^2} > \sqrt{x^2 + (y+1)^2}.$$

This inequality is satisfied if and only if

$$(y-1)^2 > (y+1)^2$$

which means that

$$|y - 1| > |y + 1|\,.$$

That is, y is closer to -1 than to 1. This condition implies that $y < 0$ and, consequently, $\|z - i\| > \|z + i\|$ corresponds to the lower half plane, $\{z : \mathrm{Im}(z) < 0\}$.

Example 135 Describe the region of the complex plane where $\|z-i\| > \|z+1\|$.

If $z = x+iy$, we have $z-i = x+i(y-1)$ and $z+1 = (x+1)+iy$. Therefore, $\|z-i\| > \|z+1\|$ is equivalent to

$$\sqrt{x^2+(y-1)^2} > \sqrt{(x+1)^2+y^2}.$$

Taking squares,

$$x^2+y^2-2y+1 > x^2+2x+1+y^2$$

which is satisfied if and only if $-2y > 2x$, that is, if

$$-y > x \Longrightarrow y < -x.$$

Since the equation $y = -x$ is the straight line of slope -1 passing through the origin $(0,0)$, $y < -x$ is satisfied for all the points (x, y) below this line.

7.2 De Moivre's formula

De Moivre's formula (also known as De Moivre's Theorem) is a consequence of Euler's formula. For any integer n, by Euler's formula, we have

$$(\mathrm{e}^{i\theta})^n = (\cos\theta + i\sin\theta)^n.$$

But, on the other hand

$$(\mathrm{e}^{i\theta})^n = \mathrm{e}^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

Therefore,

Proposition 136 (De Moivre's formula) *Let $n \in \mathbb{Z}$. Then,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{7.6}$$

This relatively simple equality has some non-trivial applications.

7.2.1 Multiple angle formulas

Equation (7.6) can be used to obtain multiple angle formulas for $\cos(n\theta)$ and $\sin(n\theta)$ by expanding its left hand side and imposing that the real and imaginary part of both sides of (7.6) have to be the same. That is,

$$\begin{aligned} \operatorname{Re}\left((\cos\theta + i\sin\theta)^n\right) &= \cos(n\theta) \\ \operatorname{Im}\left((\cos\theta + i\sin\theta)^n\right) &= \sin(n\theta). \end{aligned} \tag{7.7}$$

Example 137 Let $n = 3$. Then, by the binomial theorem,

$$\begin{aligned} (\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos(\theta)(i\sin\theta)^2 + 3\cos^2(\theta)(i\sin\theta) + (i\sin\theta)^3 \\ &= \left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right). \end{aligned}$$

Therefore, by (7.7),

$$\begin{aligned} \cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin(3\theta) &= 3\cos^2\theta\sin\theta - \sin^3\theta. \end{aligned}$$

7.3 Roots

In this section, we are going to prove that, for any $n \in \mathbb{N}$, a complex number $z \in \mathbb{C}$ has n different nth roots. Indeed, let $z = r\,\mathrm{e}^{i\theta} \in \mathbb{C}$ be a complex number in exponential notation. A complex number $w = \rho\,\mathrm{e}^{i\varphi}$ is a nth root of z if it satisfies

$$w^n = z. \tag{7.8}$$

This equation can means that

$$\rho^n\,\mathrm{e}^{in\varphi} = r\,\mathrm{e}^{i\theta}\,. \tag{7.9}$$

Comparing the modulus and the argument of both sides of this equality, we realise that, in order that (7.9) be true, we must have

$$\begin{aligned} \rho^n &= r \\ n\varphi &= \theta + 2\pi k, \quad k \in \mathbb{Z}. \end{aligned}$$

These equations, in turn, imply

$$\rho = \sqrt[n]{r} \text{ and } \varphi = \frac{\theta}{n} + \pi\frac{2k}{n},\ k \in \mathbb{Z}.$$

However, observe that, when k runs over the set of integers,

$$\mathrm{e}^{i\left(\frac{\theta}{n}+\pi\frac{2k}{n}\right)} = \cos\left(\frac{\theta}{n} + \pi\frac{2k}{n}\right) + i\sin\left(\frac{\theta}{n} + \pi\frac{2k}{n}\right)$$

can only have n different values for $k = 0, 1, ..., n-1$. In conclusion, there are n different complex numbers w_k, $k = 0, ..., n-1$, such that $w_k^n = z$. They are given by

$$w_k = \sqrt[n]{r}\,\mathrm{e}^{i\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)}, \quad k = 0, 1, ..., n-1. \tag{7.10}$$

Example 138 Suppose we want to find all complex solutions of

$$w^n = 1.$$

Since $1 = \mathrm{e}^{i0}$, according to (7.10),

$$w_k = \sqrt{1}\,\mathrm{e}^{i\left(\frac{0}{n}+\frac{2\pi k}{n}\right)} = \mathrm{e}^{i\frac{2\pi k}{n}} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \quad k = 0, 1, ..., n-1,$$

are the n different solutions. They are called the nth roots of the unity.

Equation (7.8) is a particular example of a polynomial (in w) of degree n that has n different roots. The Fundamental Theorem of Algebra states that this is true for any arbitrary polynomial.

Theorem 139 (Fundamental Theorem of Algebra) *Let $p(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then $p(x)$ has n (complex) roots.*

7.4 Hyperbolic trigonometric functions

We have seen that there is a close relationship between trigonometric functions and Euler's formula. In this subsection, we will explore this relationship further.

Writing Euler's formula for θ and $-\theta$ we get

$$\begin{aligned} \mathrm{e}^{i\theta} &= \cos\theta + i\sin\theta, \\ \mathrm{e}^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta. \end{aligned}$$

We can rearrange these two equations to obtain

$$\cos\theta = \frac{\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}}{2i}. \tag{7.11}$$

This also provides a connection between standard trigonometric functions and hyperbolic functions. Recall that the hyperbolic functions are defined as

$$\cosh y = \frac{\mathrm{e}^{y} + \mathrm{e}^{-y}}{2} \quad \text{and} \quad \sinh y = \frac{\mathrm{e}^{y} - \mathrm{e}^{-y}}{2}.$$

If z is a purely imaginary number, i.e. of the form $z = iy$, $y \in \mathbb{R}$, one may define $\cos(z)$ and $\sin(z)$ using the power series expansions for the trigonometric functions. However, using (7.11),

$$\cos(iy) = \frac{\mathrm{e}^{i^2 y} + \mathrm{e}^{-i^2 y}}{2} = \frac{\mathrm{e}^{-y} + \mathrm{e}^{y}}{2} = \cosh y$$

and

$$\sin(iy) = \frac{\mathrm{e}^{i^2 y} - \mathrm{e}^{-i^2 y}}{2i} = \frac{\mathrm{e}^{-y} - \mathrm{e}^{y}}{2i} = -\frac{1}{i}\sinh(y) = i\sinh(y).$$

Example 140 Find all the complex solutions to the equation $\tan z = 2i$.

Using the definitions of sin and cos in terms of the exponential function, we have

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}.$$

Consequently, the equation $\tan z = 2i$ reduces to

$$\frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{\mathrm{e}^{iz} + \mathrm{e}^{-iz}} = 2i^2 = -2.$$

To find the solutions to this equation we can multiply numerator and denominator by e^{iz} to get

$$\frac{\mathrm{e}^{2iz} - 1}{\mathrm{e}^{2iz} + 1} = -2,$$

which is equivalent to $\mathrm{e}^{2iz} - 1 = -2\,\mathrm{e}^{2iz} - 2 \Longrightarrow 3\,\mathrm{e}^{2iz} = -1$, which gives

$$\mathrm{e}^{2iz} = -\frac{1}{3}.$$

The easiest way to solve this last equation is to write $-1/3$ in exponential form. Since $-1/3$ lies on the negative real axis,

$$-\frac{1}{3} = \frac{1}{3}\left(\cos\left((2n+1)\pi\right) + i\sin\left((2n+1)\pi\right)\right) = \frac{1}{3}\,\mathrm{e}^{i(2n+1)\pi}$$

for any $n \in \mathbb{Z}$. Moreover, letting $z = x + iy$ we have $\mathrm{e}^{2iz} = \mathrm{e}^{2ix}\,\mathrm{e}^{-2y}$ and

$$\mathrm{e}^{2ix}\,\mathrm{e}^{-2y} = \frac{1}{3}\,\mathrm{e}^{i(2n+1)\pi}\,. \tag{7.12}$$

If now we impose that the modulus of the left and right hand side of (7.12) has to be the same, we obtain

$$\mathrm{e}^{-2y} = \frac{1}{3} \Longrightarrow 2y = \ln 3 \Longrightarrow y = \frac{\ln 3}{2}.$$

On the other hand, comparing the arguments,

$$x = (2n+1)\frac{\pi}{2}.$$

In conclusion, all complex solutions to the equation $\tan z = 2i$ are the complex number of the form

$$z = (2n+1)\frac{\pi}{2} + i\frac{\ln 3}{2}.$$

Bibliography

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