MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE
PROBLEM SHEET 9 SOLUTIONS
GRAPHIC REPRESENTATION OF FUNCTIONS

1.
$$Ax + \frac{B}{x+3} = \frac{Ax^2 + 3Ax + B}{x+3}$$

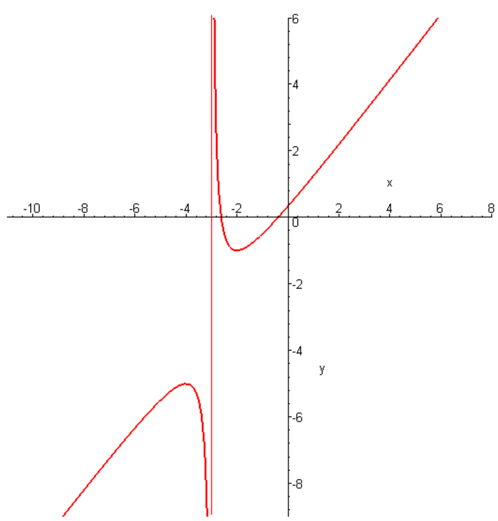
therefore $A = 1$ and $B + 1$ and
$$f(x) = x + \frac{1}{x+3}.$$

Thus
$$f'(x) = 1 - \frac{1}{(x+3})^2$$

and the stationary points are given by
$$0 = (x+3)^2 - 1 = x^2 + 6x + 8 = (x+2)(x+4),$$

so $x = -2, -4$. The second derivative is
$$f''(x) = \frac{2}{(x+3)^3}.$$

Therefore, $f''(-2) > 0$ and $x = 2$ is a minimum, and $f''(-4) < 0$ and $x = -4$ is a maximum. Vertical asymptote at $x = -3$. As $x \to \pm\infty$ we have $f(x) \approx x$ so we have the oblique asymptote $y = x$.



2.

$$f(x) = \frac{x(x+1)}{x-2} = x + 3 + \frac{6}{x-2}$$

$$f'(x) = \frac{(2x+1)(x-2) - (x^2+x)}{(x-2)^2} = \frac{x^4 - 4x - 2}{(x-2)^2}$$

So $f'(x) = 0$ if and only if $x = 2 \pm \sqrt{6}$.

$$f(2 \pm \sqrt{6}) = \frac{(2 \pm \sqrt{6})(3 \pm \sqrt{6})}{\sqrt{6}} = 5 \pm \sqrt{24}.$$

$f' > 0$ for $x < 2 - \sqrt{6}$ and $x > 2 + \sqrt{6}$ and $f' < 0$ for $2 - \sqrt{6} < x < 2 + \sqrt{6}$ except for at the vertical asymptote at $x = 2$. Thus f has a maximum at $2 - \sqrt{6}$ and a minimum at $2 + \sqrt{6}$.

The is a vertical asymptote at $x = 2$ such that

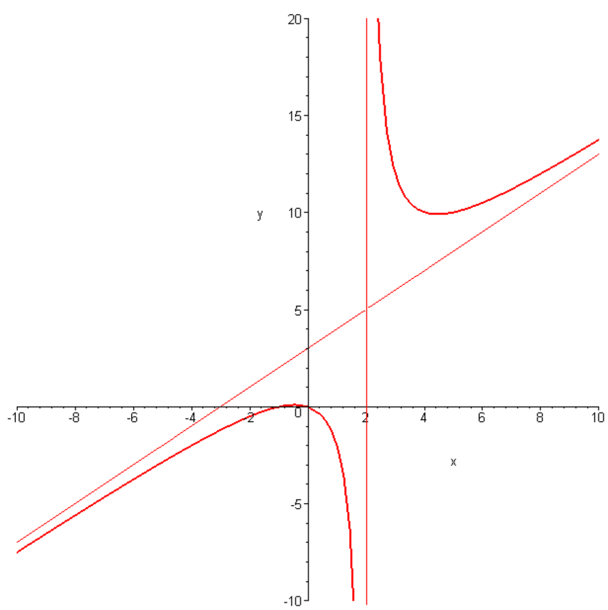
$$\lim_{x \to 2^-} \frac{x(x+1)}{x-2} = -\infty \quad \text{and} \quad \lim_{x \to 2^+} \frac{x(x+1)}{x-2} = \infty.$$

The slope m and the constant term n of the oblique asymptote are determined as

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x+1}{x-2} = 1$$

$$n = \lim_{x\to\infty} (f(x) - x) = \lim_{x\to\infty} \left(\frac{x(x+1)}{x-2} - x \right) = \lim_{x\to\infty} \frac{3x}{x-2} = 3$$

so $y = x + 3$ is the asymptote as $x \to \infty$. These limits give the same values if we replace $x \to \infty$ with $x \to -\infty$ so $y = x + 3$ is also an oblique asymptote as $x \to -\infty$.



3.

(a) $f(x) = 0$ if and only if $x^2 - 4 = 0$ or $e^{-x} = 0$. The second equation is never satisfied so the solutions are just $x = \pm 2$.

(b) f is defined everywhere and so has no vertical asymptotes. As $x \to +\infty$ we have $f(x) \to 0$ because the exponential term dominates the polynomial one, so f has a horizontal asymptote at $y = 0$. As $x \to -\infty$, $f(x) \to \infty$ so no horizontal asymptote.

(c) $f < 0$ for $-2 < x < 2$ and $f > 0$ for $x < -2$ and $x > 2$.

(d) $f'(x) = 2xe^{-x} - (x^2 - 4)e^{-x} = e^{-x}(2x - x^2 + 4)$. So

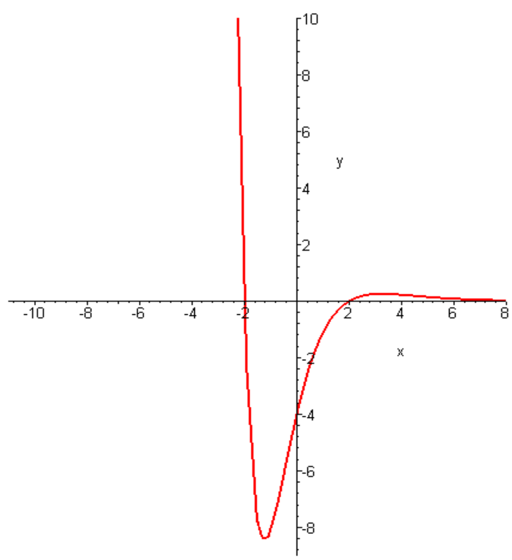
$$f'(x) = 0 \Leftrightarrow x^2 - 2x - 4 = 0 \Leftrightarrow x = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}.$$

(e) $f'(x) < 0$ for $x < 1 - \sqrt{5}$ and $x > 1 + \sqrt{5}$ and $f'(x) > 0$ for $1 - \sqrt{5} < x < 1 + \sqrt{5}$. Therefore

$$x = 1 - \sqrt{5} = \text{ local minimum: } f(1 - \sqrt{5}) = (6 - 2\sqrt{5})e^{-1+\sqrt{5}}$$

$$x = 1 + \sqrt{5} = \text{ local maximum: } f(1 + \sqrt{5}) = (6 + 2\sqrt{5})e^{-1-\sqrt{5}}.$$

(f) The graph is as follows:



4.

(a) $f(x) = 0$ implies $(x^2 + x - 2) \equiv (x + 2)(x - 1) = 0$ and so $x = -2, +1$.

(b) No vertical asymptotes. Horizontal asymptote at 0 with $f(x) \to 0^+$ (tending to zero from "above") as $x \to +\infty$.

(c) Since $f(x)$ changes sign at $x = -2$ and $x = +1$ ONLY, we have $f(x) > 0$ if $x < -2$ or $1 < x < \infty$ and $f(x) < 0$ if $-2 < x < 1$.

(d) $f'(x) = [(2x+1) - 2(x^2+x-2)]e^{-2x} = (5-2x^2)e^{-2x} = 0$ when $x = \pm\sqrt{5/2}$.

(e) Using (c) or calculating f'' we have a minimum at $x = -\sqrt{5/2}$ and a maximum at $x = +\sqrt{5/2}$.

5.

(a) **Domain.** The denominator never vanishes but $\mathrm{e}^{1/x}$ is not defined at $x = 0$. Therefore

$$D = \mathbb{R}\backslash\{0\}.$$

(b) **Intersection points.** Since

$$f(x) = \frac{x+2}{\mathrm{e}^{\frac{1}{x}}+1}$$

is not defined at $x = 0$, the graph does not intersect the y-axis. We must however evaluate the limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$.

$$\lim_{x\to 0^+} \frac{x+2}{\mathrm{e}^{\frac{1}{x}}+1} = \frac{2}{\mathrm{e}^{\infty}+1} = \frac{2}{\infty} = 0,$$
$$\lim_{x\to 0^-} \frac{x+2}{\mathrm{e}^{\frac{1}{x}}+1} = \frac{2}{\mathrm{e}^{-\infty}+1} = \frac{2}{0+1} = 2.$$

On the other hand,

$$f(x) = 0 \Longrightarrow x+2 = 0 \Longrightarrow x = -2$$

and the graph of f intersects the x-axis at $(-2,0)$.

(c) **Relative extremes.** We need to compute the derivative of $f(x)$,

$$f'(x) = \frac{\mathrm{e}^{\frac{1}{x}}+1-(x+2)\,\mathrm{e}^{\frac{1}{x}}\left(-x^{-2}\right)}{\left(\mathrm{e}^{\frac{1}{x}}+1\right)^2} = \frac{\mathrm{e}^{\frac{1}{x}}\left(1+\frac{x+2}{x^2}\right)+1}{\left(\mathrm{e}^{\frac{1}{x}}+1\right)^2}.$$

Nevertheless, one can check that

$$1+\frac{x+2}{x^2}$$

in the numerator is always positive so $f'(x) > 0$ and there are no relative extremes.

(d) **Monotonicity.** Since $f'(x) > 0$ the function is always increasing.

(e) **Asymptotes.**

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x}\frac{x+2}{\mathrm{e}^{\frac{1}{x}}+1} = \lim_{x\to\infty} \frac{x+2}{x} \lim_{x\to\infty} \frac{1}{\mathrm{e}^{\frac{1}{x}}+1} = 1\cdot\frac{1}{\mathrm{e}^0+1} = \frac{1}{2}.$$

Therefore, there exists an oblique asymptote with slope $\frac{1}{2}$. Its constant term is given by

$$\lim_{x\to\infty}\left(f(x)-\frac{1}{2}x\right) = \lim_{x\to\infty}\left(\frac{x+2}{\mathrm{e}^{\frac{1}{x}}+1}-\frac{1}{2}x\right) = \lim_{x\to\infty}\frac{-x\,\mathrm{e}^{\frac{1}{x}}+x+4}{2\left(\mathrm{e}^{\frac{1}{x}}+1\right)} = \frac{\infty-\infty}{4}. \qquad (1)$$

The indeterminate form $\infty-\infty$ appears, so the limit

$$\lim_{x\to\infty}\left(x - x\,\mathrm{e}^{\frac{1}{x}}\right)$$

has to be dealt with separately.

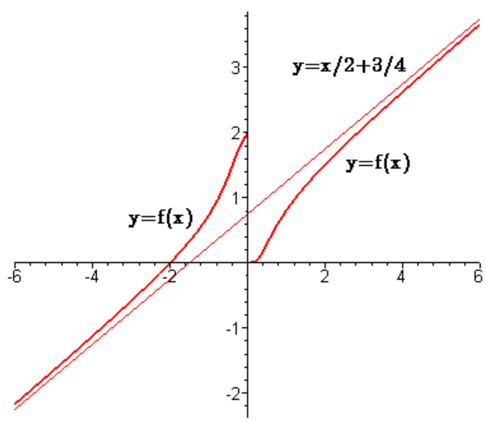
$$\lim_{x\to\infty}\left(x - x\,\mathrm{e}^{\frac{1}{x}}\right) = \lim_{x\to\infty} x\left(1-\mathrm{e}^{\frac{1}{x}}\right) = \lim_{x\to\infty}\frac{1-\mathrm{e}^{\frac{1}{x}}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{-\mathrm{e}^{\frac{1}{x}}\left(-x^{-2}\right)}{-x^{-2}}$$
$$= \lim_{x\to\infty} -\mathrm{e}^{\frac{1}{x}} = -\mathrm{e}^{0} = -1. \qquad (2)$$

Hence, from (1),

$$\lim_{x\to\infty}\frac{-x\,\mathrm{e}^{\frac{1}{x}}+x+4}{2\left(\mathrm{e}^{\frac{1}{x}}+1\right)} = \frac{\lim_{x\to\infty}\left(x-x\,\mathrm{e}^{\frac{1}{x}}\right)+4}{2\left(\mathrm{e}^{0}+1\right)} = \frac{-1+4}{4} = \frac{3}{4}$$

In conclusion, the straight line $y=\frac{1}{2}x+\frac{3}{4}$ is the oblique asymptote as $x\to\infty$. It can be checked that $y=\frac{1}{2}x+\frac{3}{4}$ is also an asymptote at $-\infty$ evaluating the previous limits when $x\to-\infty$.

(f) **Graph.**



6.

(a) **Domain.** $D=(0,\infty)$ because the logarithm of a negative number does not exist.

(b) **Intersection points.** $f(x)=0$ implies $\ln(10x)=0$. Since $x=1$ is the unique number such that $\ln(x)=0$, then $f(x)=0 \Longrightarrow x=1/10$.

(c) **Relative extremes.**

$$f'(x) = \frac{\frac{1}{x}x-\ln(10x)}{x^2} = \frac{1-\ln(10x)}{x^2} = 0 \Longrightarrow \ln(10x)=1, \qquad (3)$$

that is

$$f'(x)=0 \Longrightarrow x=\frac{\mathrm{e}}{10}.$$

Rewrite

$$f'(x)=\frac{N(x)}{D(x)}. \qquad (4)$$

Then

$$sign\left(f''(x)|_{x=\frac{e}{10}}\right) = sign\left(N'(x)|_{x=\frac{e}{10}}\right) = sign\left(-\frac{1}{10x}\bigg|_{x=\frac{e}{10}}\right) < 0$$

and we have a maximum at $(\frac{e}{10}, f(\frac{e}{10}) = \frac{10}{e})$.

(d) **Monotonicity.** Using (3) and the notation introduced in (4),

	$(0, \frac{e}{10})$	$(\frac{e}{10}, \infty)$
$sign(N)$	+	−
$sign(D)$	+	+
$sign(f')$	+	−

so the function is increasing on $(0, \frac{e}{10})$ and decreasing on $(\frac{e}{10}, \infty)$.

(e) **Asymptotes.** There is a vertical asymptote at $x = 0$,

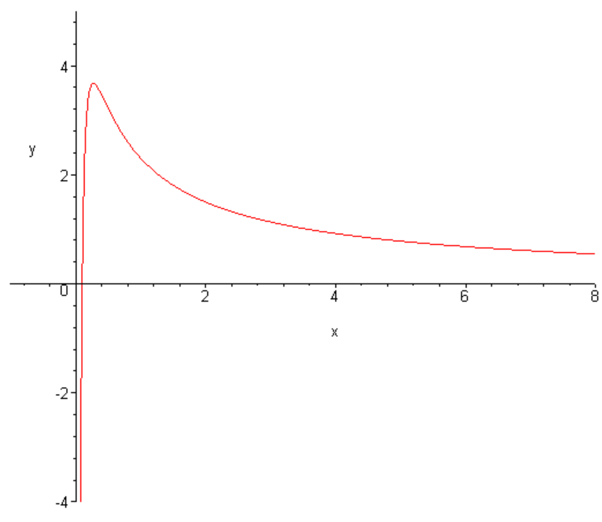
$$\lim_{x\to 0^+} \frac{\ln(10x)}{x} = \frac{-\infty}{0} = -\infty.$$

Since

$$\lim_{x\to\infty} \frac{\ln(10x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0,$$

$y = 0$ is the horizontal asymptote.

(f) **Graph.**



7.

$$f(x) = \frac{4x-5}{x(x-1)} = \frac{5}{x} - \frac{1}{x-1},$$

so $A = 5$ and $B = 1$. In order to find the stationary points we need

$$f'(x) = -\frac{5}{x^2} + \frac{1}{(x-1)^2}.$$

Setting $f'(x)$ equal to zero gives stationary points at

$$\frac{5}{4} \pm \frac{\sqrt{5}}{4}.$$

Checking the sign of the derivative on either side of the stationary points gives $\frac{5}{4} - \frac{\sqrt{5}}{4}$ local minimum and $\frac{5}{4} + \frac{\sqrt{5}}{4}$ local maximum. On the other hand, $f(x) = 0$ at $x = 5/4$, vertical asymptote at $x = 0$ and $x = 1$, horizontal asymptote at $y = 0$.