MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE
PROBLEM SHEET 8 SOLUTIONS
LIMITS

1. (*a*) We apply l'Hôpital's rule

$$\lim_{x\to\pi/4} \frac{-\sin 2x}{\sec^2 x} = \frac{-2}{(2/\sqrt{2})^2} = -1.$$

(*b*) Denominator not zero, so limit is 0.

(*c*) l'Hôpital's rule

$$\lim_{x\to 2} \frac{\frac{1}{2}(x+2)^{-1/2}}{\frac{1}{2}(x^3-4)^{-1/2}3x^2} = \frac{1}{12}.$$

(*d*) Let $y = ((x+3)/x)^x$ so that for x large enough we can write

$$\ln y = \ln(1+\frac{3}{x}) = x(\frac{3}{x} - \frac{9}{2x^2} + \ldots) = 3 - \frac{9}{2x} + \ldots.$$

Therefore $\lim_{n\to\infty} \ln y = 3$ and $\lim_{x\to\infty} y = e^3$.

2. (*a*) This limit is not any of the indeterminate forms. Therefore

$$\lim_{x\to 1} \frac{(x-2)\,(x+2)}{(x-3)\,(x+1)} = \frac{-3}{-4} = \frac{3}{4}.$$

(*b*) By l'Hôpital's rule

$$\lim_{x\to 0} \frac{1-\cos x}{\tan^2 x} = \lim_{x\to 0} \frac{\sin x}{2\tan x \sec^2 x} = \lim_{x\to 0} \frac{\cos x}{2\sec^4 x + 2\tan x (d(\sec^2 x)/dx)} = \frac{1}{2+0} = \frac{1}{2}$$

(*c*) Let $y = x^x$, then $\ln y = x \ln x$. As $x \to 0$, $\ln y \to 0$ hence $y \to 1$, i.e. $\lim_{x\to 0} x^x = 1$.

(*d*)

$$\lim_{x\to -2} \frac{\sqrt{-2x}-2}{x+2} = \lim_{x\to -2} \frac{(\sqrt{-2x}-2)(\sqrt{-2x}+2)}{(x+2)(\sqrt{-2x}+2)}$$
$$= \lim_{x\to -2} \frac{-2x-4}{(x+2)(\sqrt{-2x}+2)} = \lim_{x\to -2} \frac{-2}{\sqrt{-2x}+2} = -\frac{1}{2}.$$

3. (*a*)

$$\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} = \lim_{\theta\to 0} 2\frac{\sin 2\theta}{2\theta} = 2 \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} = 2$$

(*b*)

$$\lim_{\theta\to 0} \frac{(\sin\theta)^2}{\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \lim_{\theta\to 0} \sin\theta = 1 \cdot 0 = 0.$$

(*c*)

$$\lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta\sin\theta} = \lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta^2} \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1.$$

4. (a) Using $|x\sin(1/x)| \leq |x|$ we obtain $\lim_{x\to 0} x\sin(1/x) = 0$.

(b) Using

$$\sqrt{1+x}-\sqrt{x} = (\sqrt{1+x}-\sqrt{x})\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} = \frac{1}{\sqrt{1+x}+\sqrt{x}}$$

we obtain

$$\lim_{x\to\infty}\left(\sqrt{1+x}-\sqrt{x}\right) = 0.$$

(c) $\lim_{x\to\infty}\sin(1/x) = \lim_{y\to 0}\sin y = 0$.

(d) l'Hôpital's rule gives

$$\lim_{x\to 1}\frac{x-1}{x^n-1} = \lim_{x\to 1}\frac{1}{nx^{n-1}} = 1/n.$$

(e) Using

$$\frac{2^n}{n!} = \frac{2\cdots 2\cdots 2}{n(n-1)\cdots 1} = \frac{2}{n}\frac{2}{n-1}\cdots\frac{2}{2}\frac{1}{1} \leq \frac{4}{n}$$

since every term between the first and the last term is ≤ 1. Therefore we have

$$\lim_{n\to\infty}\frac{2^n}{n!} = 0.$$

(f) $\lim_{x\to 0}\sin(1/x)$ does not exist as the function takes every value between -1 and 1 infinitely often on every interval containing 0.

5. (a) There is no indeterminate form here, so

$$\lim_{x\to 0}\frac{x^2-1}{x^2+x-2} = \frac{-1}{-2} = \frac{1}{2}.$$

(b)

$$\lim_{x\to 1}\frac{x^2-1}{x^2+x-2} = \lim_{x\to 1}\frac{(x-1)\,(x+1)}{(x-1)(x+2)} = \lim_{x\to 1}\frac{(x+1)}{(x+2)} = \frac{2}{3}.$$

(c) Applying l'Hôpital's rule,

$$\lim_{x\to\pi/2}\sec^2(x)(1-\sin(x)) = \lim_{x\to\pi/2}\frac{1-\sin(x)}{\cos^2(x)} = \lim_{x\to\pi/2}\frac{-\cos(x)}{-2\cos(x)\sin(x)} = \lim_{x\to\pi/2}\frac{1}{2\sin(x)} = \frac{1}{2}.$$

(d) First,

$$x\left(\sqrt{x^2+4}-x\right) = x^2\left(\sqrt{1+\frac{4}{x^2}}-1\right)$$

and then we expand $\sqrt{1+\frac{4}{x^2}}$ using the binomial theorem

$$\sqrt{1+\frac{4}{x^2}} = 1+\frac{2}{x^2}-\frac{2}{x^4}+....$$

Therefore

$$\lim_{x\to\infty}x^2\left(\sqrt{1+\frac{4}{x^2}}-1\right) = \lim_{x\to\infty}x^2\left(\frac{2}{x^2}-\frac{2}{x^4}+...\right) = \lim_{x\to\infty}\left(2-\frac{2}{x^2}+...\right) = 2.$$

6. (*a*)

$$\lim_{x\to 0^+}\frac{\ln(x+1)}{\sin(x)}=\lim_{x\to 0^+}\frac{\frac{1}{x+1}}{\cos(x)}=\lim_{x\to 0^+}\frac{1}{(1+x)\cos(x)}=\frac{1}{(1+0)\cos(0)}=1$$

(*b*) We rewrite $(\sin(x))^x = \mathrm{e}^{x\ln(\sin x)}$ so that

$$\lim_{x\to 0^+}(\sin(x))^x=\lim_{x\to 0^+}\mathrm{e}^{x\ln(\sin x)}=\mathrm{e}^{\lim_{x\to 0^+}x\ln(\sin x)}$$

but

$$\lim_{x\to 0^+}x\ln(\sin x)=\lim_{x\to 0^+}\frac{\ln(\sin(x))}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{\cos(x)}{\sin(x)}}{-x^{-2}}=\lim_{x\to 0^+}\frac{-x^2}{\tan(x)}$$
$$=\lim_{x\to 0^+}\frac{-2x}{1+\tan^2(x)}=0,$$

where we have applied l'Hôpital's rule twice. Therefore

$$\lim_{x\to 0^+}(\sin(x))^x=\mathrm{e}^{\lim_{x\to 0^+}x\ln(\sin x)}=\mathrm{e}^0=1.$$

(*c*)

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2\sqrt{x}}{x}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0.$$

(*d*)

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan^{-1}(x)}\right)=\lim_{x\to 0^+}\frac{\tan^{-1}(x)-x}{x\tan^{-1}(x)}=\lim_{x\to 0^+}\frac{\frac{1}{1+x^2}-x}{\tan^{-1}(x)+\frac{x}{1+x^2}}=\frac{1}{0^+}=\infty.$$

(*e*) We rewrite $x^{\sin x}=\mathrm{e}^{\sin(x)\ln(x)}$. Then

$$\lim_{x\to 0^+}x^{\sin x}=\lim_{x\to 0^+}\mathrm{e}^{\sin(x)\ln(x)}=\mathrm{e}^{\lim_{x\to 0^+}\sin(x)\ln(x)}$$

and

$$\lim_{x\to 0^+}\sin(x)\ln(x)=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sin(x)}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{\cos(x)}{\sin^2(x)}}=\lim_{x\to 0^+}-\frac{\sin^2(x)}{x\cos(x)}$$
$$=\lim_{x\to 0^+}-\frac{\sin(x)}{x}\lim_{x\to 0^+}\frac{\sin(x)}{\cos(x)}=-1\cdot 0=0.$$

Therefore,

$$\lim_{x\to 0^+}x^{\sin x}=\mathrm{e}^{\lim_{x\to 0^+}\sin(x)\ln(x)}=\mathrm{e}^0=1.$$

(*f*)

$$\lim_{x\to\infty}\frac{x+\ln(x)}{x\ln(x)}=\lim_{x\to\infty}\left(\frac{1}{\ln(x)}+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{1}{\ln(x)}+\lim_{x\to\infty}\frac{1}{x}=0.$$