

MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE.
PROBLEM SHEET 7
POWER SERIES

1. Expand the function

$$f(x) = \frac{1}{1-x}$$

around $x = 0$ in a series up to the first four non-zero terms. Use this expansion to find an approximation for $f(x)$ at $x = 0.1$. Using the remainder term, what error bound can you give for the difference between the exact value of $f(x)$ at $x = 0.1$ and the approximation computed from the series expansion?

2. Show that $\ln(1+x)$ has a Maclaurin expansion

$$x - \frac{x^2}{2} + \frac{x^3}{3} + R_4$$

and find the form of the remainder R_4 for this function. Use the first three terms of the above expansion to find an approximate value for

$$\int_0^1 \frac{\ln(x+1)}{x} dx$$

and use the remainder R_4 to give a bound for the error.

- 3.

- (a) Explain what it means for a series

$$\sum_{n=1}^{\infty} a_n$$

to be convergent.

- (b) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

- (c) Thus, show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges as well.

(d) Find the radius of convergence R of

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}.$$

4. A function $y(x)$ satisfies the equations

$$\frac{d^2y}{dx^2} + xy = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = 0$.

(a) Differentiate the equation n times to show that, for $n \geq 1$,

$$\frac{d^{n+2}y}{dx^{n+2}} + n \frac{d^{n-1}y}{dx^{n-1}} = 0 \quad \text{at } x = 0.$$

(b) Hence show that the solutions of the equation with the given initial conditions has the form

$$y(x) = \sum_{i=0}^{\infty} c_i x^{3i}$$

where c_i are constants. Find the first three non-zero terms in this expansion and show by the ratio test that this series converges for all x .

5. A function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} - xy = 0$$

with initial conditions $y(0) = 0$, $y'(0) = 1$.

(a) Differentiate the equation $n + 1$ times to show that for $n \geq 1$

$$y^{(n+3)}(0) = (n + 1)^2 y^{(n)}(0).$$

(b) Deduce that the solution of the equation with the given initial conditions has the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{3n+1}.$$

Use the ratio test to show that this series converges for all x .

6. A function satisfies the equation

$$\frac{d^2y}{dx^2} + x^2y = 0$$

with initial conditions $y(0) = 1$, $y'(0) = 0$.

- (a) Differentiate the equation $m + 2$ times using Leibniz to show that for $m \geq 0$

$$y^{(m+4)}(0) = -(m+2)(m+1)y^{(m)}(0).$$

- (b) Deduce that the solution of the equation with the given initial conditions has the form

$$y(x) = \sum_{m=0}^{\infty} c_m x^{4m}$$

where c_m are constants. Show that this series converges for any x .

7. The function $f(x)$ is defined as

$$f(x) = (1 - x^2)^{\frac{1}{3}}.$$

- (a) Calculate the derivative $f'(x)$ and show that $f(0) = 0$.
(b) Calculate the second derivative $f''(x)$ and show that f satisfies the differential equation

$$(1 - x^2) f'' - \frac{4}{3} x f' + \frac{2}{3} f = 0.$$

Use the Leibniz formula to differentiate this equation n times and prove that

$$f^{(n+2)}(0) = \left(n^2 + \frac{n}{3} - \frac{2}{3} \right) f^{(n)}(0) \text{ for } n \geq 0.$$

- (c) Find the first three non-zero terms in the Maclaurin expansion for $f(x)$. Use the binomial expansion to check your result.