MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE
PROBLEM SHEET 7 SOLUTIONS
POWER SERIES

1. Let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$. We have

$$f'(x) = (1-x)^{-2}, \; f^{(2)}(x) = 2(1-x)^{-3}, \; f^{(3)}(x) = 6(1-x)^{-4}, \; f^{(4)}(x) = 24(1-x)^{-5}$$

so

$$f(x) = \sum_{i=0}^{3} \frac{f^{(i)}(0)}{i!} x^i + R(0;x) = 1 + x + x^2 + x^3 + R(0;x)$$

where

$$R(0;x) = \frac{f^{(4)}(x_0)}{4!} x^4 = \frac{x^4}{(1-x_0)^5}$$

where x_0 is between 0 and x. Therefore

$$f(0.1) \approx 1 + 0.1 + 0.1^2 + 0.1^3 = 1.111$$

In order to give an error bound, we need to evaluate $R(0;0.1)$ with some $x_0 \in [0, 0.1]$. This error is bigger when $x_0 = 0.1$, so the error bound is

$$\left. \frac{0.1^4}{(1-x_0)^5} \right|_{x_0=0.1} = 1.7\ 10^{-4}$$

and

$$f(0.1) - 1.111 \leq 1.7\ 10^{-4}.$$

2. For $f = \ln(1+x)$, we have

$$f'(x) = \frac{1}{1+x}, \; f^{(2)}(x) = -\frac{1}{(1+x)^2}, \; f^{(3)}(x) = \frac{2}{(1+x)^3}, \; f^{(4)}(x) = \frac{-6}{(1+x)^4}$$

so

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + R_4,$$

where

$$R_4 = -\frac{1}{4(1+x_0)^4} x^4$$

for some x_0 between 0 and x. Then

$$\begin{aligned} \int_0^1 \frac{\ln(1+x)}{x} dx &\approx \int_0^1 \frac{1}{x} \left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \right] dx = \int_0^1 \left[1 - \frac{1}{2}x + \frac{1}{3}x^2 \right] dx \\ &= \left[x - \frac{x^2}{4} + \frac{x^3}{9} \right]_0^1 = 1 - \frac{1}{4} + \frac{1}{9} = \frac{31}{36}. \end{aligned}$$

The error in the Taylor expansion is $R_4 = -\frac{1}{4(1+x_0)^4} x^4$, where $x_0 \in [0,1]$. Therefore, R_4 is a function bounded by

$$-\frac{1}{4}x^4 \leq R_4 \leq -\frac{1}{4 \cdot 2^4} x^4.$$

Multiplying by $\frac{1}{x}$ and taking integrals to this inequality,

$$-\frac{1}{4} \int_0^1 x^3 dx \leq \int_0^1 \frac{R_4}{x} dx \leq -\frac{1}{4 \cdot 2^4} \int_0^1 x^3 dx$$

The greatest error possible is given by $-\frac{1}{4}\int_0^1 x^3dx$, which is greater than $-\frac{1}{4\cdot 2^4}\int_0^1 x^3dx$ in absolute value. Then

$$-\int_0^1 \frac{1}{x}\frac{1}{4}x^4dx = -\int_0^1 \frac{x^3}{4}dx = -\left[\frac{x^4}{16}\right]_0^1 = -\frac{1}{16}$$

and we conclude that

$$\frac{31}{36} - \frac{1}{16} \leq \int_0^1 \frac{\ln(1+x)}{x}dx \leq \frac{31}{36}.$$

3.

(a) A series is convergent if the limit of partial sums $S_n = a_1 + \cdots a_n$ exists and is finite.

(b) Use the integral test,

$$\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty} f(k)$$

with $f : [x, \infty) \to \mathbb{R}$ given by $f(x) = 1/x$. Since

$$\lim_{n\to\infty} \int_1^n \frac{1}{x}dx = \lim_{n\to\infty} [\ln x]_1^n = \lim_{n\to\infty} n = \infty,$$

the series diverges.

(c) By the comparison test, $\sum_{1/\sqrt{n}}$ is divergent because $\frac{1}{\sqrt{n}} > \frac{1}{n}$ for ever $n \geq 2$.

(d) Applying the ratio test we get

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}x^{n+1}}{n+1} \frac{n}{(-1)^{n+1}x^n} \right| = |x|.$$

So $R = 1$.

4.

(a) We have $y'' + xy = 0$. Differentiating n times by Leibniz we have

$$y^{(n+2)} + xy^{(n)} + ny^{(n-1)} = 0$$

that reduces to

$$y^{(n+2)}(0) + ny^{(n-1)}(0) = 0 \quad \text{at } x = 0. \qquad (\spadesuit)$$

(b) For $n = 1$, $(\spadesuit)$ reads

$$y^{(3)}(0) = y(0) = 0 \Longrightarrow y^{(3)}(0) = -1.$$

For $n = 2$,

$$y^{(4)}(0) + ny'(0) = 0 \Longrightarrow y^{(4)}(0) = 0.$$

For $n = 3$,

$$y^{(5)}(0) + 3y''(0) = 0 \Longrightarrow y^{(5)}(0) = 0.$$

For $n = 4$,

$$y^{(6)}(0) = 4y^{(3)}(0) = 0 \Longrightarrow y^{(6)}(0) = 4.$$

The only non-zero terms are $y(0), y^{(3)}(0), y^{(6)}(0), \ldots$ and so on. The Maclaurin series is

$$y(x) = 1 - \frac{x^3}{3!} + \frac{4x^6}{6!} + \cdots + \frac{y^{(3n)}(0)x^{3n}}{(3n)!} + \ldots$$

From $(\spadesuit)$, we have $y^{(3n)}(0) + (3n-2)y^{(3n-2)}(0) = 0$, $y^{(3n)}(0)/y^{(3n-3)}(0) = -(3n-2)$, and therefore

$$\lim_{n\to\infty} \left| \frac{y^{(3n)}(0)x^{3n}}{(3n)!} \frac{(3n-3)!}{y^{(3n-3)}(0)x^{3n-3}} \right| = \lim_{n\to\infty} \left| \frac{(3n-2)x^3}{(3n)(3n-1)(3n-2)} \right| = 0$$

By the ratio test, the series converges for any x.

5.

(a) In order to establish the relationship for $y^{(n+3)}(0)$ from

$$y^{(2)} - x^2 y^{(1)} - xy = 0 \tag{1}$$

we have to, first, differentiate (1) $n+1$ times so that

$$y^{(n+3)} - \left(x^2 y^{(1)}\right)^{(n+1)} - (xy)^{(n+1)} = 0 \tag{2}$$

and, second, evaluate (2) at $x = 0$.
By the Leibniz formula we have

$$(x^2 y')^{(n+1)} = \sum_{i=0}^{n+1} \binom{n+1}{i} (x^2)^{(i)} y^{(n+2-i)}.$$

When $x = 0$, the only non-zero term here is

$$\binom{n+1}{2} (x^2)^{(2)} y^{(n)}(0) = (n+1)n y^{(n)}(0),$$

provided $n \geq 1$ (otherwise all terms are zero). By the Leibniz formula again we have

$$(xy)^{(n+1)} = \sum_{i=0}^{n+1} \binom{n+1}{i} (x^2)^{(i)} y^{(n+1-i)}.$$

For $x = 0$, the only non-zero term is $(n+1)y^{(n)}(0)$. Hence

$$y^{(n+3)}(0) = (n+1)^2 y^{(n)}(0)$$

for $n \geq 1$.

(b) From the statement, it easy to observe that $y^{(2)}(0) = 0$ and $y^{(3)}(0) = y(0) = 0$ by differentiating the equation once. Therefore, we deduce that

$$y^{(3m)}(0) = 0 \quad \text{and} \quad y^{(3m+2)}(0) = 0.$$

The series expansion of $y\,(x)$ is then

$$\sum_{m=0}^{\infty} y^{(3m+1)}(0) \frac{x^{3m+1}}{(3m+1)!} \quad \text{with } y^{(3(m+1)+1)}(0) = (3m+2)^2 y^{(3m+1)}(0).$$

Applying the ratio test

$$\lim_{m\to\infty} \frac{|x|^3\,(3m+2)^2}{(3m+4)(3m+3)(3m+2)} = 0$$

and the series converges for any x.

6.

(a) Differentiating the equation we have

$$y^{(2)} + x^2 y = 0, \quad y^{(3)} + 2xy + x^2 y' = 0 \quad \text{and} \quad y^{(4)} + 2y + 4xy' + x^2 y'' = 0.$$

In general, applying Leibniz's rule,

$$y^{(n+2)} + \sum_{i=0}^{n} \binom{n}{i} (x^2)^{(i)} y^{(n-1)} = 0.$$

For $n \geq 2$ this reduces to

$$y^{(n+2)} + \frac{n(n-1)}{2} 2y^{(n-2)} + n2xy^{(n-1)} + x^2 y^{(n)} = 0.$$

For $x = 0$ we get $y(0) = 1$, $y'(0) = 0$, $y^{(2)}(0) = 0$, $y^{(3)}(0) = 0$, $y^{(4)}(0) = -2y(0) = -2$. If $n \geq 2$, we have

$$y^{(n+2)}(0) + n(n-1)y^{(n-2)}(0) = 0.$$

(b) The non-zero terms are $y^{(0)}(0), y^{(4)}(0), \ldots, y^{(4m)}(0), ..$ and so the Maclaurin series is

$$\sum_{m=0}^{\infty} c_m x^m = 1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} + \cdots + \frac{y^{(4m)}(0) x^{4m}}{(4m)!} + \cdots$$

By the ratio test

$$R^{-1} = \lim_{m \to \infty} \left| \frac{y^{(4m)}(0)}{(4m)!} \frac{(4(m-1))!}{y^{(4(m-1))}(0)} \right| = \lim_{m \to \infty} \frac{1}{4m(4m-1)} = 0.$$

Thus the series converges for any x.

7.

(a)

$$f'(x) = -\frac{2}{3} x (1 - x^2)^{-2/3} \quad \text{and} \quad f'(0) = 0$$

(b)

$$f''(x) = -\frac{2}{9}(x^2 + 3)(1 - x^2)^{-5/3}$$

$$(1-x^2)f'' - \frac{4}{3}xf' + \frac{2}{3}f = -\frac{2}{9}(x^2+3)(1-x^2)^{-2/3} + \frac{8}{9}x^2(1-x^2)^{-2/3} + \frac{2}{3}(1-x^2)(1-x^2)^{-2/3} = 0.$$

Note that $(1 - x^2)' = -2x$ vanishes at $x = 0$, so

$$((1-x^2)f'')^{(n)}(0) = f^{(n+2)}(0) - 2\binom{n}{2} f^{(n)}(0).$$

Additionally,

$$((1-x^2)f')^{(n)}(0) = \frac{4}{3} n f^{(n)}(0),$$

so

$$f^{(n+2)}(0) = n(n-1)f^{(n)}(0) + \frac{4}{3}nf^{(n)}(0) - \frac{2}{3}f^{(n)}(0) = (n^2 + \frac{1}{3}n - \frac{2}{3})f^{(n)}(0).$$

(c) It follows that $f^{(m)}(0) = 0$ if m is odd so

$$f(x) = 1 + \frac{1}{2}f''(0)x^2 + \frac{1}{24}f^{(4)}(0)x^4 + \ldots = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 \ldots$$

The binomial expansion gives

$$(1-x^2)^{1/3} = 1 + \frac{1}{3}(-x^2) + \frac{1}{2}\frac{1}{3}(\frac{1}{3} - 1)(-x^2)^2 + \ldots = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 \ldots$$