MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE.
PROBLEM SHEET 6
THE RIEMANN INTEGRAL. RADIUS OF CONVERGENCE

1. Evaluate the following definite integrals using algebraic substitutions:

   (a) $\int_0^1 (3x+4)^{-3}\,dx$ (b) $\int_0^1 (1-x)^{1/2}\,dx$ (c) $\int_1^2 x\left(1+x^2\right)^{1/2}dx$

   (d) $\int_0^{\pi} \cos(4x)\,dx$ (e) $\int_0^{\pi/2} \sin^2(x)\cos(x)dx$

2. Evaluate the following Riemann integrals

   (a) $\int_1^2 \frac{x^2}{1+x^3}dx$ (b) $\int_0^1 x\tan^{-1}(x)dx$ (c) $\int_0^{\pi/2}\left(\sin^3(x)-3\right)\cos(x)\,dx$ (c) $\int_0^{\infty} x^5\,\mathrm{e}^{-x^2}\,dx.$

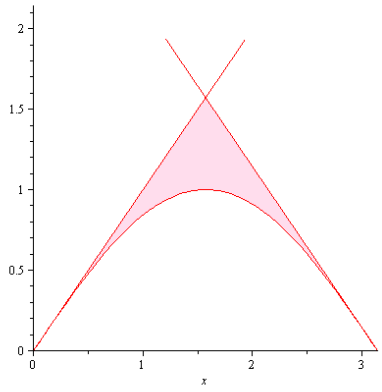
   (Hint: $\lim_{x\to\infty} x^n\,\mathrm{e}^{-x^2}=0$, $n\in\mathbb{N}$).

3. (a) Let
$$I_m=\int_0^{\pi}\cos^m(x)dx,\quad n=0,1,2,...$$
Derive a formula relating I_m and I_{m-2} (for $m\geq 2$). Use this formula to calculate I_4.

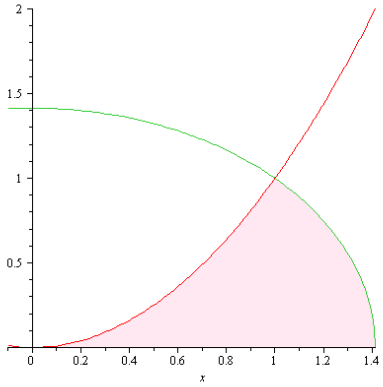
   (b) Given that
$$I_n=\int_0^{\pi}\mathrm{e}^x\sin^n(x)dx,\quad n=0,1,2,...$$
show that
$$\left(n^2+1\right)I_n=n\,(n-1)\,I_{n-2}.$$
Verify that
$$I_5=\frac{3}{13}\left(\mathrm{e}^{\pi}+1\right).$$

   (c) Evaluate the integrals
$$\int_0^{\infty}\mathrm{e}^{-x}\cos(nx)\,dx \quad\text{and}\quad \int_0^{\infty}\mathrm{e}^{-x}\sin(nx)\,dx$$
in terms of n, where n is a positive integer. Show that
$$\lim_{n\to\infty}\int_0^{\infty}\mathrm{e}^{-x}\cos(nx)\,dx=\lim_{n\to\infty}\int_0^{\infty}\mathrm{e}^{-x}\sin(nx)\,dx=0.$$

4. Calculate the area delimited by the graph of $y=\sin(x)$ and the tangents to this graph at $x=0$ and $x=\pi$.

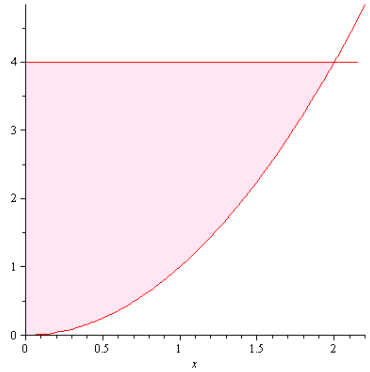


5. Calculate the area delimited by the x-axis, a circle of radius $\sqrt{2}$ and the parabola $y = x^2$ in the first quadrant.



6. Calculate the volume of the solid of revolution generated by $y = x^2$, $x \in [0, 2]$, when revolving around the x-axis.

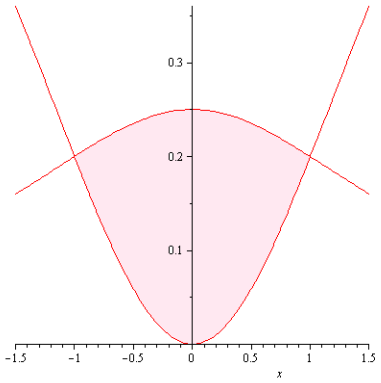
7. Calculate the volume of the solid of revolution generated by the surface delimited by $y = x^2$, $y = 4$, and the vertical y-axis when revolving around the x-axis.



8. Let

$$f(x) = \frac{1}{4+x^2} \quad \text{and} \quad g(x) = \frac{x^2}{4+x^2}.$$

Calculate the area of the surface delimited by the graphs of these two functions. If this surface revolve around the y-axis, calculate the volume of the solid of revolution thus obtained.



9. Decide whether each of the following series is convergent or divergent:

$$(a) \quad \sum_{n\geq 1} \frac{2n+5}{100n} \qquad (b) \quad \sum_{n\geq 1} \frac{n^{100}}{2^n+5}.$$

10. Obtain the radius of convergence for each of the following series:

(a) $\sum_{n=0}^{\infty} 3^n x^n$ $\quad$ (b) $\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$ $\quad$ (c) $\sum_{n=0}^{\infty} n x^n$

(d) $\sum_{n=0}^{\infty} \frac{n^2}{2^n} (x-1)^n$ $\quad$ (e) $\sum_{n\geq 1} \frac{|x-2|^n}{5^n}$ $\quad$ (f) $\sum_{n\geq 1} \frac{(n+2)!}{(2n)!} x^n$.