MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE
PROBLEM SHEET 6 SOLUTIONS
THE RIEMANN INTEGRAL. RADIUS OF CONVERGENCE

1. (*a*) Use $u = 3x + 4$:

$$\int_0^1 (3x+4)^{-3}\,dx = \int_4^7 \frac{1}{3u^3}du = -\frac{1}{6}\left[u^{-2}\right]_4^7 = \frac{1}{6}\left(\frac{1}{16} - \frac{1}{49}\right) = \frac{11}{1568}.$$

(*b*) Use $u = 1 - x$:

$$\int_0^1 \left(1-x^2\right)^{1/2} dx = \int_0^1 u^{1/2} du = \frac{2}{3}\left[u^{3/2}\right]_0^1 = \frac{2}{3}.$$

(*c*) Use $u = 1 + x^2$:

$$\int_1^2 x\left(1+x^2\right)^{1/2} dx = \frac{1}{2}\int_2^5 u^{1/2} du = \left[\frac{1}{3}u^{3/2}\right]_2^5 = \frac{\sqrt{125}-\sqrt{8}}{3}.$$

(*d*) Use $u = 4x$:

$$\int_0^\pi \cos(4x)\,dx = \frac{1}{4}\int_0^{4\pi} \cos(u)\,du = \frac{1}{4}\left[\sin u\right]_0^{4\pi} = 0.$$

(*e*) Use $x = \sin(u)$:

$$\int_0^{\pi/2} \sin^2(x)\cos(x)\,dx = \int_0^1 u^2 du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}.$$

2. (*a*)

$$\int_1^2 \frac{x^2}{1+x^3}dx = \frac{1}{3}\int_1^2 \frac{3x^2}{1+x^3}dx = \frac{1}{3}[\ln(1+x^3)]_1^2 = \frac{1}{3}[\ln 9 - \ln 2] = \frac{1}{3}\ln\frac{9}{2}$$

(*b*) (Integration by parts)

$$\int_0^1 x\tan^{-1} x dx = [\frac{x^2}{2}\tan^{-1} x]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx$$
$$= \frac{1}{2}\tan^{-1} 1 - \frac{1}{2}[x - \tan^{-1} x]_0^1 = \tan^{-1} 1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}.$$

(*c*) Let $t = \sin x$ and $dt = \cos x dx$. Then

$$\int_0^{\pi/2} (\sin^3 x - 3)\cos x dx = \int_0^1 t^3 - 3dt = [\frac{t^4}{4} - 3t]_0^1 = (\frac{1}{4} - 3) - 0 = -\frac{11}{4}.$$

(*d*) Use integration by parts $\int udv = uv - \int vdu$ with

$$v = e^{-x^2}, dv = -2xe^{-x^2} \quad \text{and} \quad u = -x^4/2, du = -2x^3.$$

We have

$$I_5 = \int_0^\infty x^5 e^{-x^2} dx = \left[-\frac{x^4 e^{-x^2}}{2}\right]_0^\infty + 2\int_0^\infty x^3 e^{-x^2} dx,$$

but

$$\left[-\frac{x^4 e^{-x^2}}{2}\right]_0^\infty = \lim_{t\to\infty}\left[-\frac{x^4 e^{-x^2}}{2}\right]_0^t = \lim_{t\to\infty} -\frac{t^4 e^{-t}}{2} = 0.$$

Therefore

$$I_5 = \int_0^\infty x^5 e^{-x^2} dx = 2\int_0^\infty x^3 e^{-x^2} dx = 2I_3.$$

Similarly

$$I_3 = \int_0^\infty x^3 e^{-x^2} dx = \left[-\frac{x^2 e^{-x^2}}{2}\right]_0^\infty + \int_0^\infty x^{e^{-x^2}} dx = I_1.$$

So

$$I_5 = 2I_3 = 2I_1 = 2\int_0^\infty x e^{-x^2} dx = 2\left[-\frac{e^{-x^2}}{2}\right]_0^\infty = 1.$$

3. (a) For $m \geq 2$ we write $(\cos x)^m = \cos x (\cos x)^{m-1}$ and use integration by parts $\int_0^\pi u dv = [uv]_0^\pi - \int_0^\pi v du$ with $u = (\cos x)^{m-1}$ and $v = -\sin x$ so

$$\int (\cos x)^{m-1} \cos x dx = \sin x (\cos x)^{m-1} + (m-1)\int (\sin x)^2 (\cos x)^{m-2} dx$$

Using $(\sin x)^2 = 1 - (\cos x)^2$, we have

$$(\sin x)^2 (\cos x)^{m-2} = (\cos x)^{m-2} + (\cos x)^m.$$

Therefore,

$$\int (\cos x)^{m-1} \cos x dx = \sin x (\cos x)^{m-1} + (m-1)\int (\cos x)^{m-2} - (m-1)\int (\cos x)^m$$

and so

$$m\int_0^\pi (\cos x)^m = [\sin x (\cos x)^{m-1}]_0^\pi + (m-1)\int_0^\pi (\cos x)^{m-2} = (m-1)\int_0^\pi (\cos x)^{m-2}.$$

We can write this as

$$I_m = \frac{m-1}{m} I_{m-2} \quad \text{which implies} \quad I_4 = \frac{3}{4} I_2 = \frac{3}{4}\frac{1}{2} I_0 = \frac{3}{8}\int_0^\pi dx = \frac{3\pi}{8}.$$

(b) Integrating by parts twice we have

$$\begin{aligned} I_n &= \int_0^\pi e^x \sin^n x dx = [e^x \sin^n x]_0^\pi - n\int_0^\pi e^x \sin^{n-1} x \cos x dx \\ &= -n[e^x \sin^{n-1} x \cos x]_0^\pi + n\int_0^\pi e^x((n-1)\sin^{n-2} x \cos^2 x - \sin^n x) dx \\ &= n\int_0^\pi e^x((n-1)\sin^{n-2} x \cos^2 x - \sin^n x) dx. \end{aligned}$$

Now we use $\cos^2 = 1 - \sin^2$ to get

$$(n-1)\sin^{n-2} x \cos^2 x - \sin^n x = (n-1)\sin^{n-2} x (1 - \sin^2 x) - \sin^n x = (n-1)\sin^{n-2} x - n\sin^n x.$$

Therefore this gives

$$I_n = \int_0^\pi e^x \sin^n x dx = n\int_0^\pi e^x((n-1)\sin^{n-2} x - n\sin^n x) dx = n(n-1)I_{n-2} - n^2 I_n$$

Putting $n = 5$ and $n = 3$ successively we get

$$I_5 = \frac{20}{26} I_3 = \frac{20}{26}\frac{6}{10} I_1 = \frac{6}{13} I_1$$

and

$$I_1 = \int_0^\pi e^x \sin x dx = -[e^x \cos x]_0^\pi + \int_0^\pi e^x \cos x dx = e^\pi + 1 + [e^x \sin \pi]_0^\pi - I_1.$$

That is, $2I_1 = e^\pi + 1$ and $I_5 = 3(e^\pi + 1)/13$.

(c) Let

$$\mathcal{C}_n = \int_0^\infty e^{-x} \cos nx dx \quad \text{and} \quad \mathcal{S}_n = \int_0^\infty e^{-x} \sin nx dx.$$

Integrating by parts with $du = e^{-x}$ in both cases, we get

$$\mathcal{C}_n = \int_0^\infty e^{-x} \cos nx = [-e^{-x} \cos nx]_0^\infty - n \int_0^\infty e^{-x} \sin nx dx = 1 - n\mathcal{S}_n.$$

$$\mathcal{S}_n = [-e^x \sin nx]_0^\infty + n \int_0^\infty e^{-x} \cos nx dx = n\mathcal{C}_n.$$

Hence

$$\mathcal{C}_n = 1 - n^2 \mathcal{C}_n \quad \text{and so} \quad \mathcal{C}_n = \frac{1}{1+n^2} \quad \text{and} \quad \mathcal{S}_n = \frac{n}{1+n^2}.$$

It follows immediately that

$$\lim_{n\to\infty} \mathcal{C}_n = \lim_{n\to\infty} \mathcal{S}_n = 0.$$

4. The tangent line at $x = 0$ passes through $(0, \sin(0)) = (0, 0)$ and has slope $y'(0)$. Its equation is given by

$$y - 0 = y'(0)(x - 0) \Longrightarrow y = \cos(0)\, x = x.$$

The tangent at $x = \pi$ passes through $(\pi, \sin(\pi)) = (\pi, 0)$ and its slope is $\cos(\pi) = -1$. Therefore, its equation is

$$y - 0 = (-1)(x - \pi) \Longrightarrow y = \pi - x.$$

These two lines meet at $(\pi/2, \pi/2)$. The area we want to calculate is then

$$\int_0^{\pi/2} (x - \sin(x))\, dx + \int_{\pi/2}^{\pi} (\pi - x - \sin(x))\, dx.$$

On the one hand,

$$\int_0^{\pi/2} (x - \sin(x)) = \left[\frac{x^2}{2} + \cos(x)\right]_0^{\pi/2} = \frac{\pi^2}{8} - \cos(0) = \frac{\pi^2}{8} - 1.$$

On the other hand

$$\int_{\pi/2}^{\pi} (\pi - x - \sin(x))\, dx = \left[\cos(x) + \pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi} = \frac{\pi^2}{8} - 1,$$

so the total area is

$$\frac{\pi^2}{4} - 2.$$

5. The parabola $y = x^2$ and the circle $x^2 + y^2 = 2$ intersect at $(1, 1)$. From $x = 0$ to $x = 1$, the upper boundary of the surface is given by the parabola. From $x = 1$ to $x = \sqrt{2}$, the upper boundary is given by the circle. Therefore, the area is

$$\int_0^1 x^2 dx + \int_1^{\sqrt{2}} \sqrt{2 - x^2} dx.$$

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

On the other hand,

$$\int_1^{\sqrt{2}} \sqrt{2-x^2}dx = \sqrt{2}\int_1^{\sqrt{2}} \sqrt{1-\left(x/\sqrt{2}\right)^2}dx.$$

We apply the change of variables

$$\begin{aligned}\frac{x}{\sqrt{2}} &= \sin(u) \Longrightarrow dx = \sqrt{2}\cos(u)\,du \\ x &= 1 \Longrightarrow \sin(u) = \frac{1}{\sqrt{2}} \Longrightarrow u = \frac{\pi}{4} \\ x &= \sqrt{2} \Longrightarrow \sin(u) = 1 \Longrightarrow u = \frac{\pi}{2}\end{aligned}$$

and

$$\begin{aligned}\sqrt{2}\int_1^{\sqrt{2}} \sqrt{1-\left(x/\sqrt{2}\right)^2}dx &= 2\int_{\pi/4}^{\pi/2} \sqrt{1-\sin^2(u)}\cos(u)\,du \\ &= 2\int_{\pi/4}^{\pi/2} \cos^2(u)\,du = 2\int_{\pi/4}^{\pi/2} \frac{1+\cos(2u)}{2}du \\ &= \left[u+\frac{\sin(2u)}{2}\right]_{\pi/4}^{\pi/2} = \left(\frac{\pi}{4}-\frac{1}{2}\right).\end{aligned}$$

The result is

$$\frac{1}{3}+\frac{\pi}{4}-\frac{1}{2} = \frac{\pi}{4}-\frac{1}{6}.$$

6. The volume is obtained as the integral over $[0,2]$ of the area of the circular section $S(x) = \pi y^2(x) = \pi x^4$.

$$\int_0^2 \pi x^4 = \pi\left[\frac{x^5}{5}\right]_0^2 = \pi\frac{2^5}{5}.$$

7. The parabola $y = x^2$ and $y = 4$ intersect at $(2,4)$, so we will integrate over $[0,2]$. The volume of the solid of revolution generated by the upper boundary $y = 4$ is given by

$$\int_0^2 \pi y^2(x)dx = \int_0^2 \pi 4^2 dx = \pi 2^5.$$

From this volume, we have to subtract the volume of the solid of revolution generated by the lower boundary $y = x^2$,

$$\int_0^2 \pi y^2(x)dx = \int_0^2 \pi x^4 dx = \pi\frac{2^5}{5}.$$

The result is

$$\pi 2^5 - \pi\frac{2^5}{5} = \pi\frac{4}{5}2^5 = \pi\frac{2^7}{5}.$$

8. The two graphs intersect at

$$\frac{1}{4+x^2} = \frac{x^2}{4+x^2} \Longrightarrow x = \pm 1.$$

Since $f \geq g$ on $[-1,1]$, the area is given by

$$\begin{aligned}\int_{-1}^1 (f(x)-g(x))\,dx &= \int_{-1}^1 \left(\frac{1}{4+x^2}-\frac{x^2}{4+x^2}\right)dx \\ &= \int_{-1}^1 \frac{1-x^2}{4+x^2}dx = 2\int_0^1 \frac{1-x^2}{4+x^2}dx\end{aligned}$$

because the integrand is even. Since $\deg\left(1-x^2\right)=\deg\left(4+x^2\right)$ we first divide,

$$1-x^2=(-1)\left(4+x^2\right)+5$$

and now

$$\begin{aligned}
2\int_0^1 \frac{1-x^2}{4+x^2}dx &= 2\int_0^1(-1)\,dx+2\int_0^1\frac{5}{4+x^2}dx\\
&= -2+10\int_0^1\frac{1}{2^2+x^2}dx\\
&= -2+10\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^1=-2+5\tan^{-1}\left(\frac{1}{2}\right).
\end{aligned}$$

As far as the volume is concerned, since the surface is revolving around the y-axis, we need first to invert the functions f,g

$$\begin{aligned}
y &= \frac{1}{4+x^2}\Longrightarrow x=\sqrt{\frac{1-4y}{y}}\\
y &= \frac{x^2}{4+x^2}\Longrightarrow x=2\sqrt{\frac{y}{1-y}}.
\end{aligned}$$

From $y=0$ to $y=g(1)=1/5$ the boundary of the solid of revolution is given by $g(x)$ and, from $y=1/5$ to $y=f(0)=1/4$, by $f(x)$. Therefore, the volume is given by

$$\int_0^{1/5}\pi\left(g^{-1}(y)\right)^2dy+\int_{1/5}^{1/4}\pi\left(f^{-1}(y)\right)^2dy.$$

On the one hand,

$$\begin{aligned}
\int_0^{1/5}\pi\left(g^{-1}(y)\right)^2dy &= \pi\int_0^{1/5}4\frac{y}{1-y}dy=4\pi\int_0^{1/5}\left(-1+\frac{1}{1-y}\right)dy\\
&= 4\pi\left[-y-\ln(1-y)\right]_0^{1/5}=4\pi\left(-\frac{1}{5}-\ln\left(\frac{4}{5}\right)\right)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{1/5}^{1/4}\pi\left(f^{-1}(y)\right)^2dy &= \pi\int_{1/5}^{1/4}\frac{1-4y}{y}=\pi\int_{1/5}^{1/4}\left(\frac{1}{y}-4\right)dy\\
&= \pi\left[\ln y-4y\right]_{1/5}^{1/4}=\pi\left(\ln\left(\frac{1/4}{1/5}\right)-4\left(\frac{1}{4}-\frac{1}{5}\right)\right)\\
&= \pi\left(\ln\left(\frac{5}{4}\right)-\frac{1}{20}\right).
\end{aligned}$$

The total volume is

$$4\pi\left(-\frac{1}{5}-\ln\left(\frac{4}{5}\right)\right)+\pi\left(\ln\left(\frac{5}{4}\right)-\frac{1}{20}\right)=\pi\left(5\ln\left(\frac{5}{4}\right)-\frac{17}{20}\right)=0.83.$$

9. (a) Since

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{2n+5}{100n}=\frac{2}{100}\neq 0$$

the series diverge.

(b) The 5 in the denominator of $\frac{n^{100}}{2^n+5}$ plays no role. Indeed, the series given by $x_n = \frac{n^{100}}{2^n+5}$ can be dominated by $y_n = \frac{n^{100}}{2^n}$. We will prove that the latter converges. By the ratio test

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \frac{\frac{(n+1)^{100}}{2^{n+1}}}{\frac{n^{100}}{2^n}} = \lim_{n\to\infty} \frac{(n+1)^{100}}{n^{100}} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{100} \frac{1}{2} = \frac{1}{2} < 1,$$

so the series converges.

10. (a) This series is a geometric series with ratio $r = 3x$. It is convergent if $|3x| < 1$. that is, if $|x| < \frac{1}{3}$.

(b) By the ratio test,

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{|x|^{2(n+1)}}{|x|^{2n}} = \lim_{n\to\infty} \frac{1}{n+1} |x|^2 = 0 < 1$$

for any x. Therefore the radius of convergence is ∞.

(c)

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)|x|^{n+1}}{n|x|^n} = \lim_{n\to\infty} \frac{n+1}{n} |x| = |x|,$$

so the series converges if $|x| < 1$ and diverges if $|x| > 1$. For $|x| = 1$ the series is

$$\sum_{n\geq 0} n \quad \text{or} \quad \sum_{n\geq 0} n(-1)^n$$

which diverge. The radius of convergence is therefore 1.

(d)

$$\begin{aligned}
\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} &= \lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}|x-1|^{n+1}}{\frac{n^2}{2^n}|x-1|^n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{2^n}{2^{n+1}} \frac{|x-1|^{n+1}}{|x-1|^n} \\
&= \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \frac{|x-1|}{2} = \frac{|x-1|}{2}.
\end{aligned}$$

The series converges if

$$\frac{|x-1|}{2} < 1 \Longrightarrow |x-1| < 2$$

So the radius of convergence is 2.

(e) By the ratio test

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{\frac{|x-2|^{n+1}}{5^{n+1}}}{\frac{|x-2|^n}{5^n}} = \lim_{n\to\infty} \frac{5^n}{5^{n+1}} \frac{|x-2|^{n+1}}{|x-2|^n} = \lim_{n\to\infty} \frac{1}{5}|x-2| = \frac{1}{5}|x-2|.$$

$$\frac{1}{5}|x-2| < 1 \Longrightarrow |x-2| < 5.$$

The radius of convergence is 5.

(f)

$$\begin{aligned}
\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} &= \lim_{n\to\infty} \frac{\frac{(n+3)!}{(2(n+1))!}|x|^{n+1}}{\frac{(n+2)!}{(2n)!}|x|^n} = \lim_{n\to\infty} \frac{(n+3)!}{(n+2)!} \frac{(2n)!}{(2(n+1))!} \frac{|x|^{n+1}}{|x|^n} \\
&= \lim_{n\to\infty} (n+3) \frac{1}{(2n+2)(2n+1)} |x| = \lim_{n\to\infty} \frac{n+3}{4n^2+5n+2} |x| = 0
\end{aligned}$$

for any x. The radius of convergence is ∞.