MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE.
PROBLEM SHEET 4.a
DIFFERENTIAL CALCULUS

1. Use the definition of the derivative as a limit to find the derivative of $f(x) = x^n$ where n is an integer. (Hint: Assume the binomial theorem.)

2. Differentiate the following:

(a) $3x^2 + 2x - x^{-1} - 7x^{-2}$	(b) $\tan(x)$	(c) e^x
(d) $x\sin(x)$	(e) $\sin(x)/x$	(f) $e^x \cos x$
(g) $x^n \ln(x)$	(h) $\ln(1+x)$	(i) $(x^3+6)\tan x$
(j) e^{x^2+1}	(k) $\sin(x^2)$	(l) $(\sin x)^2$
(m) $\sin(2x)$	(n) $\ln(1+\sin x)$	(o) e^{kx}/x
(p) $\left(1+x^4\right)^{1/2}$	(q) $(1+2\sin(3x))^5$	(r) $\ln(x+\left[x^2+1\right]^{1/2})$

3. Find the second derivatives of:

$$3x^5 - x + 10 - 2x^{-1}, \qquad \sin(kx), \qquad e^x - e^{-x}, \qquad \ln(ax).$$

4. Find the stationary points of the following functions and determine their nature. Do the functions have other points of inflexion? Sketch the curve in each case.

(a) $x^2 - 2x + 3$	(b) $\frac{1}{3}x^3 - x + 2$	(c) $x^3 - x^2 - 4x + 4$
(d) $\sin^2 x$	(e) $x^{2/3}(1-x)$	(f) $x/\left(1+x^2\right)$

5. Prove that the Maclaurin's series of the following functions are as given:
$\sin x = x - x^3/3! + x^5/5! - ...,$
$\cos x = 1 - x^2/2 + x^4/4! - ...,$
$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4!...,$
$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + ...$ for $|x| < 1,$
$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + ...$ for $|x| < 1,$
$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...$ for $|x| < 1.$
Each result has its own characteristic pattern which makes it easy to remember. Please remember these results. (Hint: the Maclaurin's series associated to a function $f(x)$ is given by $f(0) + \sum_{n\geq 1} \frac{f^{(n)}(0)}{n!}x^n$.)

6. Find $\frac{dy}{dx}$ in the following cases. Simplify when necessary.

(a) $y = \frac{x\,e^x}{\ln x}$	(b) $y = x^{\ln x}$	(c) $x + y + e^{xy} = 1$
(d) $y = \frac{\sin x}{1+\cos x}$	(e) $y = e^{x+x^2}\cosh x$	(f) $y = \tan^{-1}(x)$
(g) $y = \frac{\sin(2x)}{x^2+2}$	(h) $y = \sinh^{-1}(\frac{3x}{4})$	(i) $y^2 = \sin(xy)$

7. Find $\frac{dy}{dx}$ in terms of t if

   (a) $x(t) = t + \sin t$ and $y(t) = t + \cos t$. Show that

$$(1+\cos t)^3 \frac{d^2y}{dx^2} = \sin t - \cos t - 1.$$

(b) $x(t) = t - \cos t$ and $y(t) = t - \sin t$. Show that

$$\frac{dy}{dx} = \tan\left(\frac{t}{2}\right).$$

(c) $x(t) = \cos^2 t$ and $y(t) = \tan t$. Express also $\frac{dy}{dx}$ in terms of x and y. Check your answer by finding a relation between x and y and then differentiating.

8. Find a general formula for $\frac{d^n y}{dx^n}$ in each of the following cases:

$$(a)\quad y = e^{2x} \qquad (b)\quad y = \ln(1-x) \qquad (c)\quad y = x^2\, e^{2x}$$

9. Consider $y(x) = e^{x^2/2}$. Show that $\frac{dy}{dx} = xy$. By differentiating this equation n times using the Leibniz's formula, show that

$$y^{(n+1)}(x) = xy^{(n)} + ny^{(n-1)}(x).$$

Evaluate $y^{(5)}(0)$.

10. A cylindrical container with closed ends is designed to have height $h = 3cm$ and radius $r = 1cm$. Due to a manufacturing error the radius becomes $1.04cm$, though the height is correct. By considering the total external surface area, including ends, as a function $A(r)$ of r, and using the formula

$$\frac{dA}{dr} = \lim_{\delta r \to 0} \frac{A(r + \delta r) - A(r)}{\delta r}$$

find the approximate increase in the surface area as a portion of the designed surface area.

11. Given

$$x^2 + y^2 + xy = 1,$$

find $\frac{dy}{dx}$. If $y = 1$ when $x = 0$, use this expression to find the approximative value of y at $x = 0.1$.

12. *Campbell's soups* wants to launch a new soup in a cylindrical can of aluminium of 0.3 litres of volume. In order to reduce costs and save as much aluminium as possible, help Campbell's determine the radius and the height of the can so that it has as less surface as possible.

13. One of the corners of our square mirror has broken as in the picture. If we want to cut it again in order to obtain a new rectangular mirror, determine x and y so that the new mirror has maximum surface.