MATHEMATICS DEPARTMENT, IMPERIAL COLLEGE
PROBLEM SHEET 4.a SOLUTIONS
DIFFERENTIAL CALCULUS

1. By definition:

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{\left(\sum_{i=0}^{n} {}^nC_i h^i x^{n-i}\right) - x^n}{h} \\ &= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2!}hx^{n-2} + ... + h^{n-1} = nx^{n-1}.\end{aligned}$$

2.

(a) $6x+2+x^{-2}+14x^{-3}$	(b) $\sec^2(x)$	(c) e^x
(d) $\sin(x)+x\cos(x)$	(e) $(x\cos x - \sin x)/x^2$	(f) $\mathrm{e}^x(\cos x - \sin x)$
(g) $x^{n-1}(x\ln x + 1)$	(h) $1/(1+x)$	(i) $3x^2\tan x + (x^3+6)\sec^2 x$
(j) $2x\,\mathrm{e}^{x^2+1}$	(k) $2x\cos\left(x^2\right)$	(l) $2\sin(x)\cos(x) = \sin(2x)$
(m) $2\cos(2x)$	(n) $\cos(x)\,/\,(1+\sin(x))$	(o) $\mathrm{e}^{kx}(kx-1)/x^2$
(p) $2x^3\left(1+x^4\right)^{-1/2}$	(q) $30\cos(3x)\,(1+2\sin(3x))^4$	(r) $\frac{1+x(x^2+1)^{-1/2}}{x+(x^2+1)^{1/2}} = (x^2+1)^{-1/2}$

3.

$$60x^3 - 4x^{-3}, \qquad -k^2\sin(kx), \qquad \mathrm{e}^x - \mathrm{e}^{-x}, \qquad -1/x^2.$$

4. Stationary points:

(a) At $x=1$, $y=2$ there is a minimum.

(b) At $x=-1$, $y=8/3$ there is a maximum, at $x=1$, $y=4/3$ there is a minimum; there is another point of inflexion at $x=0$, $y=2$.

(c) At $x=\frac{1+\sqrt{13}}{3}$ there is a minimum, at $x=\frac{1-\sqrt{13}}{3}$ there is a maximum; there is another point of inflexion at $x=1/3$, $y=2\frac{16}{27}$.

(d) There are minima at the points $x=k\pi$, $k\in\mathbb{Z}$, $y=0$ and maxima at $x=\pi/2+k\pi$, $k\in\mathbb{Z}$, $y=1$. There are other points of inflexion at $x=\pi/4+k\pi/2$, $k\in\mathbb{Z}$, $y=1/2$.

(e) At $x=2/5$ there is a maximum.

(f) At $x=1$, $y=1/2$ there is a maximum; at $x=-1$ $y=-1/2$ there is a minimum. There are other points of inflexion at $x=0$, $y=0$, $x=\sqrt{3}$, $y=\sqrt{3}/4$ and $x=-\sqrt{3}$, $y=-\sqrt{3}/4$.

5. Use Maclaurin's series:

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(0)}{2!} + x^3\frac{f'''(0)}{3!} + ... + x^n\frac{f^{(n)}(0)}{n!} + ...$$

6. (a)

$$\frac{dy}{dx} = \frac{(xe^x)' - (xe^x/x}{(\ln x)^2} = \frac{(e^x + xe^x)\ln x - e^x}{(\ln x)^2} = \frac{e^x(\ln x + x\ln x - 1)}{(\ln x)^2}.$$

(b)

$$\ln y = \ln(x)\ln(x) = (\ln x)^2 \quad \text{and so} \quad \frac{dy}{dx}\frac{1}{y} = \frac{2}{x}\ln x \Longrightarrow \frac{dy}{dx} = 2\ln(x)(x^{\ln x - 1}).$$

(c)

$$1 + \frac{dy}{dx} + (\frac{d(xy)}{dx})e^{xy} = 0 \quad \text{and so} \quad 1 + \frac{dy}{dx} + (y + x\frac{dy}{dx})e^{xy} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{ye^{xy}+1}{xe^{xy}+1}.$$

(*d*)

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos^2 x + \sin^2 x + \cos x}{(1+\cos x)^2} \\ &= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}.\end{aligned}$$

(*e*)

$$\frac{dy}{dx} = (1+2x)e^{x+x^2}\cosh x + e^{x+x^2}\sinh x = e^{x+x^2}((1+2x)\cosh x + \sinh x).$$

(*f*) By the inverse function rule,

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\tan'\left(\tan^{-1}(x)\right)}.$$

Since $\tan'(y) = 1+\tan^2(y)$,

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{\tan'\left(\tan^{-1}(x)\right)} = \frac{1}{1+\tan^2\left(\tan^{-1}(x)\right)} = \frac{1}{1+x^2}.$$

(*g*) By the quotient rule

$$\frac{dy}{dx} = \frac{2(x^2+2)\cos 2x - 2x\sin^2 x}{(x^2+2)^2}.$$

(*h*) We have $\sinh y = 3x/4$. Differentiating both sides with respect to x we get

$$\cosh y\frac{dy}{dx} = \frac{3}{4} \Longrightarrow \frac{dy}{dx} = \frac{3}{4}\frac{1}{\cosh y} = \frac{3}{4}\frac{1}{(1+\sinh^2 y)^{1/2}}.$$

Now, since $y = \sinh^{-1}(3x/4)$

$$\frac{dy}{dx} = \frac{3}{4}\frac{1}{(1+\sinh^2 y)^{1/2}} = \frac{3}{4}\frac{1}{(1+\frac{9x^2}{16})^{1/2}} = \frac{3}{(16+9x^2)^{1/2}}.$$

(*i*) Differentiating both sides of $y^2 = \sin(xy)$,

$$\frac{d}{dx}y^2 = \frac{d}{dx}\sin(xy) \Longrightarrow 2y\frac{dy}{dx} = \cos(xy)[y + x\frac{dy}{dx}]$$

which implies

$$\frac{dy}{dx} = \frac{y\cos(xy)}{2y - x\cos(xy)}.$$

7.

(*a*)

$$\frac{dx}{dt} = 1+\cos t \quad \text{and} \quad \frac{dy}{dt} = 1-\sin t \Longrightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-\sin t}{1+\cos t}.$$

We can rewrite this equality as

$$(1+\cos t)\frac{dy}{dx} = 1-\sin t.$$

Differentiating both sides with respect to t gives

$$-\sin t\frac{dy}{dx} + (1+\cos t)\frac{d^2y}{dx^2}\frac{dx}{dt} = -\cos t$$

and therefore

$$(1+\cos t)^2\frac{d^2y}{dx^2}=\sin t\left(\frac{1-\sin t}{1+\cos t}\right)-\cos t=\frac{\sin t-\sin^2 t-\cos t-\cos^2 t}{1+\cos t}.$$

Finally,

$$(1+\cos t)^3\frac{d^2y}{dx^2}=\sin t-\cos t-1.$$

(*b*)

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{1-\cos t}{\sin t}=\frac{2\sin^2(t/2)}{2\sin(t/2)\cos(t/2)}=\tan(t/2).$$

(*c*) Since $x(t)=\cos^2 t$ and $y(t)=\tan t$,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sec^2 t}{-2\cos t\sin t}=-\frac{1}{2\cos^3 t\sin t}=\frac{-1}{2\cos^4 t\tan t}=\frac{-1}{2x^2y}.$$

On the other hand, from $\tan^2 t=1+\sec^2 t$ we see that $y^2=1+\frac{1}{x}$. Therefore $2y\frac{dy}{dx}=-1/x^2$ and $dy/dx=-1/2x^2y$.

8.

(*a*) $y'=2e^{2x}$, $y''=2^2e^{2x}$, ..., $y^{(n)}=2^ne^{2x}$.

(*b*) $y'=-(1-x)^{-1}$, $y''=-(1-x)^{-2}$, $y'''=-2(1-x)^{-3}$, $\ldots$, $y^{(n)}=-(n-1)!(1-x)^{-n}$.

(*c*) Use Leibniz's formula (formulae sheet) with $f=x^2$ and $g=e^{2x}$. We have that $f^{(n)}=0$ for any $n\geq 3$ and $g^{(n)}=2^ne^{2x}$ by (*a*). Thus

$$\begin{aligned}\frac{d^n}{dx^n}(x^2e^{2x})&=fg^{(n)}+nf'g^{(n-1)}+\frac{n(n-1)}{2!}f''g^{(n-1)}+\cdots\\&=x^22^ne^{2x}+n2x2^{n-1}e^{2x}+\frac{n(n-1)}{2}2\cdot 2^{n-2}e^{2x}\\&=(2^nx^2+n2^nx+n(n-1)2^{n-2})e^{2x}.\end{aligned}$$

9. By the chain rule $y'=(2x/2)e^{x^2/2}=xy$. By Leibniz' rule ($f(x)=x$, $g(x)=y(x)$, $f^{(n)}=0$ for any $n\geq 2$),

$$y^{(n+1)}=xy^{(n)}+{}^nC_1y^{(n-1)}+0=xy^{(n)}+ny^{(n-1)}.$$

Putting $x=0$ gives $y^{(n+1)}=ny^{(n-1)}$ so that, for $n=4$, we have

$$y^{(5)}(0)=4y^{(3)}(0)=8y'(0)=0.$$

10. Since $A=2\pi r^2+2\pi r\cdot 3=2\pi(r^2+3r)$,

$$\delta A=A(r+\delta r)-A(r)\approx\frac{dA}{dr}\delta r=2\pi(2r+3)\delta r$$

therefore

$$\frac{\delta A}{A}\approx\frac{2r+3}{r^2+3r}\delta r=\frac{5}{4}0.04=0.05.$$

11. Differentiating $x^2+y^2+xy=1$ with respect to x,

$$2x+2y\frac{dy}{dx}+y+x\frac{dy}{dx}=0\Longrightarrow\frac{dy}{dx}=-\frac{2x+y}{x+2y}.$$

Then $y(x+\delta x)-y(x)\approx y'(x)\delta x$. For $x=0$, $\delta x=0.1$ we have $y'(0)=-1/2$. Consequently

$$y(0.1)\approx y(0)+y'(x)\delta x=1-.05=0.95.$$

12. The volume of a cylindrical can is $V = \pi r^2 h$, where r is the radius of the basis and h is the height. The surface is given by $A = 2\pi rh + 2\pi r^2$ (two circles and the lateral surface). Since $V = 0.3l = 3/10\ dm^3$, $h = 3/(10\pi r^2)\ dm$ and

$$A(r) = 2\pi r \frac{3}{10\pi r^2} + 2\pi r^2 = \frac{3}{5}\frac{1}{r} + 2\pi r^2.$$

Now,

$$A'(r) = -\frac{3}{5}\frac{1}{r^2} + 4\pi r = 0 \Longrightarrow 4\pi r = \frac{3}{5}\frac{1}{r^2} \Longrightarrow r = \sqrt[3]{\frac{3}{20\pi}} = 0.363\ dm = 3.63\ cm.$$

Since

$$A''(r)|_{r=0.363} = \left.\frac{6}{5}\frac{1}{r^3} + 4\pi\right|_{r=0.363} > 0$$

the function $A(r)$ has a minimum at $r = 0.363$.

13. The surface of the new mirror will be $A = (80 - x)(80 - y)$. However, x and y are related by

$$\tan\alpha = \frac{32}{40} = \frac{y}{40 - x} \Longrightarrow y = \frac{160 - 4x}{5}.$$

Therefore,

$$\begin{aligned} A(x) &= (80 - x)\left(80 - \frac{160 - 4x}{5}\right). \\ A'(x) &= -\left(80 - \frac{160 - 4x}{5}\right) + (80 - x)\frac{4}{5} = -\frac{8}{5}x + 16 \end{aligned}$$

so that

$$A'(x) = 0 \Longrightarrow x = 10cm.$$

Since

$$A''(x)|_{x=10} = \left.-\frac{8}{5}\right|_{x=10} < 0 \Longrightarrow x = 10 \text{ is a maximum.}$$