

# WEIGHT CYCLING AND SERRE-TYPE CONJECTURES FOR UNITARY GROUPS

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ABSTRACT. We prove that for forms of  $U(3)$  which are compact at infinity and split at places dividing a prime  $p$ , in generic situations the Serre weights of a mod  $p$  modular Galois representation which is irreducible when restricted to each decomposition group above  $p$  are exactly those previously predicted by the third author. We do this by combining explicit computations in  $p$ -adic Hodge theory, based on a formalism of strongly divisible modules and Breuil modules with descent data which we develop in the paper, with a technique that we call “weight cycling”.

## 1. INTRODUCTION

The weight part of (generalisations of) Serre’s conjecture has received considerable attention in recent years. There have been many new conjectures formulated (cf. [BDJ10], [Her09], [Sch08b], [Gee10a]) and several cases of these conjectures have been established (cf. [Gee10b], [GS10a]). However, there have been essentially no theoretical results for groups of semisimple rank greater than one. In this paper we prove the first such results, by showing (under suitable genericity hypotheses) that for a Galois representation which is irreducible when restricted to each decomposition group above  $p$ , its Serre weights on forms of  $U(3)$  which are compact at infinity and split at places dividing  $p$ , and for which it is modular, are precisely those predicted by [Her09]. (The same set of weights is also predicted to be a subset of the modular weights by Doud’s modification [Dou07] of the conjecture of Ash, Doud, Pollack, and Sinnott [ADP02], [AS00].) We remark that other results have recently been obtained in [GG09] (in the ordinary case), and the forthcoming [BLGG11].

We now explain our results in more detail. Let  $F$  be an imaginary CM field in which  $p$  splits completely, and  $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_p)$  be a continuous irreducible representation. Let  $F^+$  be the maximal totally real subfield of  $F$ , and let  $G$  be a unitary group over  $F^+$  which is isomorphic to  $U(3)$  at each infinite place and split at each prime above  $p$ . In Definition 7.3.5 we define what it means for  $\bar{r}$  to be modular for  $G$  (of some specific weight);

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roughly speaking, it means that the characteristic polynomials of Frobenius elements at unramified places should correspond to Hecke polynomials of some automorphic Hecke eigenform on  $G$ . This implies that  $\bar{r}$  is essentially conjugate self-dual. We have a notion of a (strongly) generic Serre weight (see Definition 7.5.4), and we let  $W_{\text{gen}}(\bar{r})$  be the set of generic Serre weights for which  $\bar{r}$  is modular. We also define  $W^?( \bar{r} )$ , the set of conjectural Serre weights for  $\bar{r}$ , following the recipe of [Her09]. (See Subsection 7.5.) Then our main theorem (Theorem 7.5.6 in the body of the text) is:

**Theorem 1.** *Suppose that  $\bar{r} : G_F \rightarrow \text{GL}_3(\overline{\mathbb{F}}_p)$  is a continuous representation. For all places  $w|p$  of  $F$  suppose that  $\bar{r}|_{G_{F_w}}$  is irreducible. If  $\bar{r}$  is modular of some strongly generic Serre weight, then  $W_{\text{gen}}(\bar{r}) = W^?( \bar{r} )$ .*

This result may be regarded as a generalisation of the results of [Gee10b] and [GS10a] to 3-dimensional representations, but the methods of proof are rather different, and in particular make no use of automorphy lifting theorems. The argument naturally breaks up into two parts. We first prove that any generic weight for which  $\bar{r}$  is modular is one predicted by [Her09]. We then prove that all these predicted weights actually appear.

The first part of the argument makes use of calculations in  $p$ -adic Hodge theory. While in essence these are of a similar nature to those of [Gee10b] and [Sch08d] (which in spirit go back at least as far as [Edi92]), they are rather more involved than those earlier calculations, for several reasons. Firstly, the representation theory of  $\text{GL}_3(\mathbb{F}_p)$  is significantly more complicated than that of  $\text{GL}_2(\mathbb{F}_p)$ , which makes the combinatorics rather more involved. More significantly, the calculations of [Gee10b] and [Sch08d] are for potentially crystalline Galois representations with Hodge–Tate weights in  $[0, 1]$ , whose reductions mod  $p$  correspond to finite flat group schemes with descent data. In our setting we must work instead with 3-dimensional potentially crystalline Galois representations with Hodge–Tate weights in  $[0, 2]$ , whose reductions no longer correspond to finite flat group schemes. However, thanks to the recent work of Tong Liu ([Liu08]) which established a conjecture of Breuil on the existence of strongly divisible lattices in semistable Galois representations with Hodge–Tate weights in  $[0, p - 2]$  (with no restrictions on the ramification), one knows that (the reductions mod  $p$  of) the representations we need to study will correspond to certain Breuil modules with descent data.

In fact, we are not aware of any results in the literature covering the extension of Liu’s results to incorporate coefficients and descent data, so we establish the basic results in this paper. We do not attempt a comprehensive treatment, but limit ourselves to allowing coefficients in the rings of integers of finite extensions of  $\mathbb{Q}_p$ , and consider only tame descent data. Our treatment follows that of [Sav05], which considers the case of Hodge–Tate weights in  $[0, 1]$ . It will not surprise an expert that this extension is possible, and our arguments here are of an essentially formal nature.

Having established these results, we are reduced to performing certain calculations with Breuil modules with descent data. Again, these calculations are far more delicate than those of [Gee10b] and [Sch08d], for several reasons. Firstly, we no longer have the underlying crutch of finite flat group schemes, which means that we cannot immediately make use of scheme-theoretic closure techniques, or make use of properties of étale models. Thankfully, the work of Caruso ([Car09]), and in particular his theory of maximal Breuil modules, provides an adequate substitute for our purposes. In addition, we know of no way to make use of the fact that our representations are potentially crystalline, rather than just potentially semistable, and we do not know of any way to directly transfer the information that the Hodge–Tate weights of our representations are  $0, 1, 2$  to the Breuil modules we consider. Instead, we assume only that our representations are potentially semistable with Hodge–Tate weights in  $[0, 2]$ , and use various ad hoc arguments using the assumption that our mod  $p$  representations are (locally) irreducible. We also make significant use of the determinant of the representation (in particular, this is how we exploit the fact that our Hodge–Tate weights are  $0, 1, 2$ ).

Having in this way established the “easy” direction of the weight part of Serre’s conjecture, we now tackle the “hard” direction of proving that a mod  $p$  representation that is modular of some particular weight is actually modular of all the expected weights. In contrast to [Gee10b] and [BLGG11], we do not make use of any automorphy lifting theorems to do this. Instead, we make use of a method which we call “weight cycling”. In the case of the group  $\mathrm{GL}_2/\mathbb{Q}_p$ , this technique was discovered by Kevin Buzzard, and was first written up in Section 5 of [Tay06]. The argument was extended to  $\mathrm{GL}_2/F$  for  $F$  an arbitrary finite extension of  $\mathbb{Q}_p$  in [Sch08a], and was also exploited in [GS10a]. We extend it to arbitrary split connected reductive groups over  $p$ -adic fields. The result we obtain is of the following form: if a mod  $p$  representation  $\bar{r}$  is modular of some given weight, and a certain Hecke operator at  $p$  vanishes on the corresponding cohomology class, then  $\bar{r}$  is modular of some other weight in a short list of possibilities. Using the elimination result already proved, this list can often be reduced to a single weight. The argument can then be repeated, and in this way we can “cycle” through all the conjectural weights.

Of course, in order to apply the weight cycling result we need to establish a vanishing result for certain Hecke operators in characteristic  $p$  (Corollary 4.5.3). Here we again use our assumption that our mod  $p$  Galois representation is irreducible when restricted to any decomposition group above  $p$ . We establish a rather general comparison between Hecke operators at  $p$  in characteristic zero and characteristic  $p$  (Proposition 4.4.2), and use it to show that if the relevant Hecke operators did not vanish, certain (normalised) Hecke operators in characteristic 0 would have to act via  $p$ -adic units. Using local-global compatibility one can deduce that the local Galois representation would be reducible, a contradiction. This argument may

be regarded as a generalisation of the fact that the Galois representations associated to ordinary modular forms are necessarily reducible.

We remark that while automorphy lifting theorem techniques such as those used in [BLGG11] are undeniably a powerful method for proving results about Serre weight conjectures, it does not seem to be possible to prove Theorem 1 by such methods. The problem is that these methods rely on producing automorphic Galois representations in characteristic 0, and deducing information about Serre weights by reducing modulo  $p$ . However, irreducible representations of the algebraic group  $\mathrm{GL}_3$  over  $\overline{\mathbb{F}}_p$  cannot always be lifted to characteristic 0, making the link between characteristic 0 and characteristic  $p$  far weaker than it is for  $\mathrm{GL}_2$ . Concretely, if we are working over a CM field  $F$  of degree  $2d$  over  $\mathbb{Q}$  in which  $p$  splits completely, then we generically expect to have  $9^d$  Serre weights for a given Galois representation (assumed irreducible when restricted to decomposition groups at places above  $p$ ), and in this paper we produce all  $9^d$  weights (in those cases to which our results apply). By contrast, automorphy lifting techniques appear to be limited to producing  $6^d$  weights.

In future work we hope to generalise the results of this paper. In particular, we hope to determine whether these techniques can be extended to the case of 4-dimensional representations, or to the case of 3-dimensional representations when  $p$  is unramified, but not necessarily split, in our CM field.

**1.1. The organisation of the paper.** In order to make it easier for the reader to see exactly which assumptions are used in each argument, and with an eye to future applications, we have gone to considerable lengths to axiomatise as many of our assumptions as possible, and to argue within our abstract axiomatic framework, rather than with specific unitary groups. For example, where arguments apply for a general  $\mathrm{GL}_n$ , we have presented them in this generality.

In Section 2 we recall the definitions of weights and their associated Hecke algebras, as well as the Satake isomorphism of [Her11b]. We also prove our general weight cycling result for an arbitrary split connected reductive group, and establish some special cases of the tame inertial local Langlands correspondence for  $\mathrm{GL}_n$ .

In Section 3 we establish the necessary results from  $p$ -adic Hodge theory; in particular, we generalise some of the results of [Liu08] to allow for coefficients and descent data. We then use these results to give an upper bound for the set of possible reductions mod  $p$  of 3-dimensional potentially semistable representations of  $G_{\mathbb{Q}_p}$  with Hodge–Tate weights  $0, -1, -2$  which are of particular tame types, under the assumption that these reductions are irreducible.

In Section 4 we introduce our basic axiomatic framework. We have sets of axioms dealing with both characteristic  $p$  and characteristic zero contexts, and we show how the characteristic zero axioms imply the characteristic  $p$

ones; this is an abstraction of the results on local-global compatibility and the comparison of Hecke operators in characteristics zero and  $p$  discussed above.

In Section 5 we use the calculations of Section 3 to establish the elimination result that we need; that is, to show that in generic situations, if  $\bar{r}$  is modular of some weight then that weight is one of the weights predicted in [Her09]. We carry out these arguments in the setting of Section 4.

In Section 6 we combine the results of Section 5 with our weight cycling technique and the vanishing of Hecke operators proved in Subsection 4.5 to prove an abstract version of Theorem 1 (namely Theorem 6.2.3) in our axiomatic framework.

In Section 7 we begin by recalling some standard material on automorphic forms on compact unitary groups and their associated Galois representations. We then establish that the axioms of Section 4 hold in this setting, and thus establish Theorem 1. Finally, in Subsection 7.6 we use automorphic induction and base change for unitary groups to show that there are many Galois representations which satisfy the hypotheses of our main theorem.

We refer to the beginning of each section for more details regarding its contents.

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**1.3. Notation and terminology.** For any field  $K$ , we let  $G_K$  denote an absolute Galois group of  $K$ , the precise choice of  $G_K$  depending on the choice of an algebraic closure  $\bar{K}$  of  $K$ . In the case of the field  $\mathbb{Q}_p$ , it will be convenient to fix once and for all an algebraic closure  $\bar{\mathbb{Q}}_p$ , with ring of integers  $\bar{\mathbb{Z}}_p$  and residue field  $\bar{\mathbb{F}}_p$ . If  $K$  is an algebraic extension of  $\mathbb{Q}_p$ , it will always be implicit that  $K$  is a subfield of this fixed algebraic closure  $\bar{\mathbb{Q}}_p$ , and we may then unambiguously set  $G_K := \text{Gal}(\bar{\mathbb{Q}}_p/K)$ . It is also convenient to fix once and for all an isomorphism  $\iota : \bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ , and we do so.

Let  $F$  be a finite extension of  $\mathbb{Q}_l$ , for  $l$  possibly equal to  $p$ . We let  $\text{Art}_F$  be the isomorphism  $F^\times \xrightarrow{\sim} W_F^{\text{ab}}$  of local class field theory, normalised so that geometric Frobenius elements correspond to uniformisers. Let  $\text{rec}_{F,\mathbb{C}}$  denote the local Langlands correspondence from isomorphism classes of irreducible smooth representations of  $\text{GL}_n(F)$  over  $\mathbb{C}$  to isomorphism classes of  $n$ -dimensional Frobenius semisimple Weil–Deligne representations of  $W_F$ . (See [HT01].) We define the local Langlands correspondence  $\text{rec}_F$  over  $\bar{\mathbb{Q}}_p$

by  $\iota \circ \text{rec}_F = \text{rec}_{F, \mathbb{C}} \circ \iota$ . It depends only on  $\iota^{-1}(q^{(n-1)/2})$ , where  $q$  is the cardinality of the residue field. Let  $N(\pi)$  denote the monodromy operator of  $\text{rec}_F(\pi)$ .

We let  $\overline{\mathbb{Q}}$  denote the algebraic closure of  $\mathbb{Q}$  in  $\overline{\mathbb{Q}_p}$ . All algebraic extensions  $F$  of  $\mathbb{Q}$  are implicitly understood to be subfields of  $\overline{\mathbb{Q}}$ , so that we may unambiguously define  $G_F := \text{Gal}(\overline{\mathbb{Q}}/F)$ .

Where possible, we use  $\rho$  and  $\bar{\rho}$  to denote representations of the absolute Galois groups of local fields, and  $r$  and  $\bar{r}$  to denote representations of the absolute Galois groups of number fields. We let  $\varepsilon$  denote the  $p$ -adic cyclotomic character, and we let  $\bar{\varepsilon}$  denote its reduction modulo  $p$ .

If  $K$  is a finite extension of  $\mathbb{Q}_p$  and  $\rho$  is a continuous de Rham representation of  $G_K$  over  $\overline{\mathbb{Q}_p}$ , then we will write  $\text{WD}(\rho)$  for the corresponding Weil–Deligne representation of  $W_K$  (defined, for example, in Appendix B.1 of [CDT99]), and if  $\tau : K \hookrightarrow \overline{\mathbb{Q}_p}$  then we will write  $\text{HT}_\tau(\rho)$  for the multiset of Hodge–Tate numbers of  $\rho$  with respect to  $\tau$ . By definition, if  $W$  is a de Rham representation of  $G_K$  over  $\overline{\mathbb{Q}_p}$  and if  $\tau : K \hookrightarrow \overline{\mathbb{Q}_p}$  then the multiset  $\text{HT}_\tau(W)$  contains  $i$  with multiplicity  $\dim_{\overline{\mathbb{Q}_p}}(W \otimes_{\tau, K} \widehat{K}(i))^{G_K}$ . Thus for example  $\text{HT}_\tau(\varepsilon) = \{-1\}$ . We will refer to (the isomorphism class of)  $\text{WD}(\rho)|_{I_K}$  as the *inertial type* of  $\rho$ . We will let  $\text{WD}(\rho)^{\text{F-ss}}$  denote the Frobenius semisimplification of  $\text{WD}(\rho)$ .

We will refer to the mod  $p$  Galois representations that are associated to automorphic representations as *modular*, as this seems to be standard practice in the literature; we remark that one could equally well refer to them as *automorphic*.

We also signal to the reader that we will frequently use the word *weight* to refer to an irreducible representation (or an isomorphism class thereof) of the  $k$ -valued points of a reductive group over a finite field  $k$ . (See Definition 2.1.1 below.) A *Serre weight* is then a variant of this notion, defined in a suitable global context. (See Definition 7.3.1.) Hopefully no confusion will arise with the other standard use of the word “weight” in representation theory (i.e. in the sense of a character of a maximal torus).

## 2. REPRESENTATION THEORY

This section is devoted to various representation-theoretic preliminaries. In Subsection 2.1 we recall some basic terminology and facts related to weights, and in Subsection 2.2 we recall the Satake isomorphism of [Her11b]. In Subsection 2.3 we then introduce the technique of weight cycling in a general setting. Finally, in Subsection 2.4, which is in a somewhat different vein to the preceding three subsections, we establish some simple instances of the so-called inertial local Langlands correspondence for the group  $\text{GL}_n(F)$ , with  $F$  a finite extension of  $\mathbb{Q}_p$ .

**2.1. Weights.** We introduce notation that will be in force for this subsection and the two that follow. Namely, let  $F$  be a finite extension of  $\mathbb{Q}_p$

with ring of integers  $\mathcal{O}_F$ , uniformiser  $\varpi$ , and residue field  $k$  of order  $q$ . Let  $G_{/\mathcal{O}_F}$  be a split connected reductive group and fix a maximal split torus  $T_{/\mathcal{O}_F}$ . Let  $\Phi \subset X^*(T)$  denote the set of roots and choose a system of simple roots  $\Delta \subset \Phi$ . Denote by  $B_{/\mathcal{O}_F}$  denote the associated Borel subgroup and by  $U_{/\mathcal{O}_F}$  its unipotent radical. Let  $W$  be the Weyl group and  $K := G(\mathcal{O}_F)$ , a hyperspecial maximal compact subgroup of  $G(F)$ .

**Definition 2.1.1.** A *weight* is an isomorphism class of irreducible representations  $V$  of  $G(k)$  over  $\bar{k}$ .

Since  $G(k)$  is a finite group, there are only finitely many weights.

*Remark 2.1.2.* We will constantly engage in the following standard abuse of terminology: namely, we will speak of some particular  $G(k)$ -representation  $V$  being a weight, when we actually mean that  $V$  is an isomorphism class representative of a weight.

For a standard parabolic subgroup  $P = MN$  we denote by  $\bar{P} = M\bar{N}$  the opposite parabolic. The following result is Lemma 2.5 in [Her11b].

**Lemma 2.1.3.** *Suppose that  $V$  is a weight and that  $P = MN$  is a standard parabolic. Then  $V_{N(k)}^{N(k)}$  and  $V_{\bar{N}(k)}^{\bar{N}(k)}$  are weights for  $M$  and the natural,  $M(k)$ -linear map  $V_{N(k)}^{N(k)} \rightarrow V_{\bar{N}(k)}^{\bar{N}(k)}$  is an isomorphism. In particular,  $V^{U(k)} \cong V_{\bar{U}(k)}^{\bar{U}(k)}$  is one-dimensional.*

Suppose that  $\mu \in X_*(T)$ . Let  $P_\mu = M_\mu N_\mu$  denote the parabolic subgroup of  $G$  defined by  $\mu$ . The following lemma is a special case of Proposition 3.8 in [Her11b]. We use the shorthand  ${}^tK = tKt^{-1}$ .

**Lemma 2.1.4.** *Let  $t = \mu(\varpi)$ . Then the image of  $K \cap {}^tK$  in  $G(k)$  is  $P_{-\mu}(k)$ .*

**2.2. Hecke operators and the Satake isomorphism.** If  $V$  is a weight, then we define the *Hecke algebra of  $V$*  as  $\mathcal{H}_G(V) := \text{End}_{G(F)}(\text{c-Ind}_K^{G(F)} V)$ . By Frobenius reciprocity we can and usually will think of it as  $\bar{k}$ -algebra of compactly supported functions  $f : G(F) \rightarrow \text{End}_{\bar{k}} V$  satisfying  $f(k_1 g k_2) = k_1 \circ f(g) \circ k_2$  for all  $k_1, k_2 \in K, g \in G(F)$ , where the multiplication is given by convolution. Note that if  $\pi$  is a smooth  $G(F)$ -representation,  $\mathcal{H}_G(V)$  naturally acts on the left on  $(V \otimes_{\mathbb{F}_p} \pi)^K$ . Explicitly, if  $\phi \in \mathcal{H}_G(V)$  and  $x \in (V \otimes_{\mathbb{F}_p} \pi)^K$ , then

$$(2.2.1) \quad \phi \cdot x = \sum_{\gamma \in G(F)/K} (\phi(\gamma) \otimes \gamma)x.$$

We now recall some results of [Her11b]. Let  $T^-$  denote the submonoid of  $T(F)$ ,

$$T^- = \{t \in T(F) : \text{ord}_F(\alpha(t)) \leq 0 \quad \forall \alpha \in \Delta\},$$

and let  $\mathcal{H}_T^-(V^{U(k)})$  denote the subalgebra of  $\mathcal{H}_T(V^{U(k)})$  consisting of those  $\varphi : T(F) \rightarrow \bar{k}$  that are supported on  $T^-$ .

**Theorem 2.2.2.** *Suppose that  $V$  is a weight. Then*

$$\mathcal{S}_G : \mathcal{H}_G(V) \rightarrow \mathcal{H}_T(V^{U(k)})$$

$$f \mapsto \left( t \mapsto \sum_{u \in U(F)/U(\mathfrak{o}_F)} f(tu) \Big|_{V^{U(k)}} \right)$$

is an injective  $\bar{k}$ -algebra homomorphism with image  $\mathcal{H}_{\bar{T}}(V^{U(k)})$ .

In particular,  $\mathcal{H}_G(V) \cong \bar{k}[X_*(T)_-]$  is commutative and noetherian (by Jordan's lemma). Here,  $X_*(T)_- = \{\mu \in X_*(T) : \langle \mu, \alpha \rangle \leq 0 \ \forall \alpha \in \Delta\}$ . We recall that  $G(F) = \coprod K\mu(\varpi)K$ , where  $\mu$  ranges over  $X_*(T)_-$  (refined Cartan decomposition). Moreover,  $\mathcal{H}_G(V)$  has a basis given by  $\bar{T}_{\mu, \varpi}$  ( $\mu \in X_*(T)_-$ ), where  $\bar{T}_{\mu, \varpi}$  has support  $K\mu(\varpi)K$  and sends  $\mu(\varpi)$  to the endomorphism  $V \rightarrow V_{N_{\mu(k)}} \xleftarrow{\sim} V^{N_{-\mu(k)}} \hookrightarrow V$ . Note that up to scalar multiple,  $\bar{T}_{\mu, \varpi}$  is independent of the choice of uniformiser  $\varpi$ .

**2.3. Weight cycling.** The following proposition and its corollary encode the basic technique of weight cycling.

**Proposition 2.3.1.** *Suppose that  $V$  is a weight and that  $\pi$  is a smooth  $G(F)$ -representation. Suppose that  $\mu \in X_*(T)_-$  and let  $t = \mu(\varpi)$ . Then we have the following commutative diagram, where the map on the right is induced by Frobenius reciprocity and the map on the top is injective.*

$$\begin{array}{ccc} (V \otimes_{\mathbb{F}_p} \pi)^K \subset \xrightarrow{i} (\text{Ind}_{K \cap {}^t K}^K V^{N_{-\mu(k)}} \otimes_{\mathbb{F}_p} \pi)^K & & \\ & \searrow \bar{T}_{\mu, \varpi} & \downarrow \\ & & (V \otimes_{\mathbb{F}_p} \pi)^K \end{array}$$

Note that the natural map  $V^{N_{-\mu(k)}} \rightarrow V$  is  $K \cap {}^t K$ -linear by Lemma 2.1.4.

*Proof.* Suppose that  $x \in (V \otimes_{\mathbb{F}_p} \pi)^K$ . By (2.2.1) and the definition of  $\bar{T}_{\mu, \varpi}$  we see that  $\bar{T}_{\mu, \varpi} \cdot x = \sum_{K/K \cap {}^t K} (k \otimes k)y$ , where

$$y := (\bar{T}_{\mu, \varpi}(t) \otimes t)x \in (V^{N_{-\mu(k)}} \otimes_{\mathbb{F}_p} \pi)^{K \cap {}^t K}.$$

We define the map  $i$  by composing the map  $x \mapsto y$  with the isomorphism

$$(2.3.2) \quad (V^{N_{-\mu(k)}} \otimes_{\mathbb{F}_p} \pi)^{K \cap {}^t K} \xrightarrow{\sim} (\text{Ind}_{K \cap {}^t K}^K V^{N_{-\mu(k)}} \otimes_{\mathbb{F}_p} \pi)^K,$$

which holds by Frobenius reciprocity. Explicitly, the isomorphism in (2.3.2) is induced by  $V^{N_{-\mu(k)}} \otimes_{\mathbb{F}_p} \pi \rightarrow \text{Ind}_{K \cap {}^t K}^K V^{N_{-\mu(k)}} \otimes_{\mathbb{F}_p} \pi$  sending  $v \otimes z$  to  $\sum_{K/(K \cap {}^t K)} [k, v] \otimes kz$ , where  $[k, v]$  denotes the element of the induction that is supported on  $(K \cap {}^t K)k^{-1}$  and sends  $k^{-1}$  to  $v$ . Also, the natural map  $\text{Ind}_{K \cap {}^t K}^K V^{N_{-\mu(k)}} \rightarrow V$  sends  $[k, v]$  to  $kv$ . It follows that the above diagram commutes.

Let  $\eta : V \rightarrow V_{N_\mu(k)} \xrightarrow{\sim} \overline{\mathbb{F}}_p$  in  $V^\vee$ , where the identification with  $\overline{\mathbb{F}}_p$  is arbitrary. Note that  $i(x) = 0$  if and only if  $\eta(x) = 0$  in  $\pi$ . Since  $\eta$  generates  $V^\vee$  as  $K$ -representation, it follows that  $i$  is injective.  $\square$

**Corollary 2.3.3.** *With the notation as in the proposition, if  $\overline{T}_{\mu, \varpi}$  fails to be injective on  $(V \otimes_{\overline{\mathbb{F}}_p} \pi)^K$ , then  $(V' \otimes_{\overline{\mathbb{F}}_p} \pi)^K \neq 0$  for at least one of the irreducible constituents  $V'$  of  $\ker(\text{Ind}_{K \cap {}^t K}^K V^{N-\mu(k)} \rightarrow V)$ .*

**2.4. Inertial local Langlands.** The purpose of this subsection is to establish some simple instances of “inertial local Langlands” for  $\text{GL}_n(F)$ , where  $n \geq 1$  is arbitrary and  $F$  is a finite extension of  $\mathbb{Q}_p$ .

Let  $E/F$  denote the unramified extension of degree  $n$ . Let  $\mathcal{O}_E$  (resp.  $\mathcal{O}_F$ ) denote the ring of integers of  $E$  (resp.  $F$ ), and let  $k_E$  (resp.  $k_F$ ) denote the residue field of  $E$  (resp.  $F$ ). Fix an  $\mathcal{O}_F$ -basis of  $\mathcal{O}_E$ . It gives rise to a rational maximal torus  $\mathbb{T}$  of  $\text{GL}_{n/k_F}$  such that  $k_E^\times = \mathbb{T}(k_F)$ . A character  $\theta : k_E^\times \rightarrow \overline{\mathbb{Q}}_p^\times$  is said to be *primitive* if its  $\text{Gal}(k_E/k_F)$ -conjugates are all distinct. If this is the case, the Deligne–Lusztig representation  $(-1)^{n-1} R_{\mathbb{T}}^\theta$  of  $\text{GL}_n(k_F)$  over  $\overline{\mathbb{Q}}_p$  is genuine, irreducible, and cuspidal. (See Proposition 7.4 and Theorem 8.3 in [DL76].)

Let  $B_n = T_n U_n$  denote the Borel subgroup of  $\text{GL}_n$  of upper triangular matrices. Let  $I(1)$  denote the pro- $p$  Iwahori subgroup. It is the inverse image of  $U_n(k_F)$  in  $\text{GL}_n(\mathcal{O}_F)$ . We will sometimes regard  $\text{GL}_n(k_F)$ -representations as  $\text{GL}_n(\mathcal{O}_F)$ -representations via inflation. If  $\chi$  is a character of  $k_F^\times$  (respectively  $\theta$  is a character of  $k_E^\times$ ) we will write  $\chi \circ \text{Art}_F^{-1}$  (respectively  $\theta \circ \text{Art}_E^{-1}$ ) for the character of  $I_F$  given by composition with the homomorphism  $I_F \rightarrow k_F^\times$  induced by  $\text{Art}_F^{-1}$  (respectively with the homomorphism  $I_E \rightarrow k_E^\times$  induced by  $\text{Art}_E^{-1}$ ).

**Proposition 2.4.1.** *Suppose that  $\pi$  is an irreducible admissible smooth representation of  $\text{GL}_n(F)$  over  $\overline{\mathbb{Q}}_p$ .*

- (i) *If  $\pi|_{\text{GL}_n(\mathcal{O}_F)}$  contains the cuspidal representation  $(-1)^{n-1} R_{\mathbb{T}}^\theta$  for some primitive character  $\theta : k_E^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ , then*

$$\text{rec}_F(\pi)|_{I_F} \cong \bigoplus_{\sigma \in \text{Gal}(k_E/k_F)} \sigma(\theta \circ \text{Art}_E^{-1})$$

and  $N(\pi) = 0$ .

- (ii) *If  $\pi|_{\text{GL}_n(\mathcal{O}_F)}$  contains the principal series representation*

$$\text{Ind}_{B_n(k_F)}^{\text{GL}_n(k_F)} (\chi_1 \otimes \cdots \otimes \chi_n)$$

for some distinct characters  $\chi_i : k_F^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ , then  $\text{rec}_F(\pi)|_{I_F} \cong \bigoplus_{i=1}^n \chi_i \circ \text{Art}_F^{-1}$  and  $N(\pi) = 0$ .

Note that the cuspidal representation in part (i) and the principal series representation in part (ii) are irreducible. (In the latter case this is because the  $\chi_i$  are distinct.)

*Proof.* Using our isomorphism  $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$  we will think of  $\pi$ ,  $\theta$ ,  $\chi_i$ , etc., as complex representations for the remainder of this proof. Recall from §1.3 that we have  $\iota \circ \text{rec}_F = \text{rec}_{F,\mathbb{C}} \circ \iota$ .

For the first part, let  $\tau$  denote  $(-1)^{n-1} R_{\mathbb{T}}^{\theta}$ . It has central character  $\theta|_{k_F^\times}$ . Thus there is a unique extension of the inflation of  $\theta$  to  $\mathcal{O}_E^\times$  to a character  $\tilde{\theta}$  of  $E^\times$  such that  $\tilde{\theta}|_{F^\times}$  is the central character of  $\pi$ . Moreover  $\tau$  extends to a  $F^\times \text{GL}_n(\mathcal{O}_F)$ -representation by letting  $F^\times$  act via  $\tilde{\theta}$ , and Frobenius reciprocity gives a  $\text{GL}_n(F)$ -equivariant map  $\text{c-Ind}_{F^\times \text{GL}_n(\mathcal{O}_F)}^{\text{GL}_n(F)} \tau \rightarrow \pi$ . From Proposition 7.3 in [DL76] and [BH10, (2.1.1)] it follows that  $\tau$  is uniquely characterised among cuspidal representations by the identity  $\text{tr} \tau(x) = (-1)^{n-1} \sum_{\sigma \in \text{Gal}(k_E/k_F)} \theta(\sigma x)$  for all  $x \in k_E^\times$  such that all  $\text{Gal}(k_E/k_F)$ -conjugates of  $x$  are distinct. From Theorem 2 in [BH10, §2.4] we see that  $\pi \cong \text{c-Ind}_{F^\times \text{GL}_n(\mathcal{O}_F)}^{\text{GL}_n(F)} \tau$  is the supercuspidal representation that is automorphically induced from  $\mu_0 \tilde{\theta} : E^\times \rightarrow \mathbb{C}^\times$ , where  $\mu_0$  is the unramified character sending uniformisers to  $(-1)^{n-1}$ . It follows by induction on  $n$  from Lemma VII.2.6(6) in [HT01] that  $\text{rec}_{F,\mathbb{C}}(\pi) \cong \text{Ind}_{W_E}^{W_F}(\mu_0 \tilde{\theta} \circ \text{Art}_E^{-1})$ . Since  $\mu_0$  is unramified and this representation is irreducible, the claim follows.

For the second part, let  $\chi : T_n(k_F) \rightarrow \mathbb{C}^\times$  denote the character  $\chi_1 \otimes \cdots \otimes \chi_n$ . If  $I \subset \text{GL}_n(\mathcal{O}_F)$  denotes the inverse image of  $B_n(k_F)$ , then we can define a smooth character  $\rho : I \rightarrow \mathbb{C}^\times$  as the composite of the natural map  $I \twoheadrightarrow T_n(k_F)$  with  $\chi$ . Theorem 7.7 and Remark 7.8 in [Roc98] show that we have an isomorphism  $\pi^\rho \xrightarrow{\sim} \pi_{U_n}^\chi$  where  $\pi_{U_n}$  is the Jacquet module and where the superscripts on either side denote isotypic components. Our assumption, together with Frobenius reciprocity, shows that  $\pi^\rho \neq 0$ , so  $\pi_{U_n}^\chi \neq 0$ . It follows that there is a  $T_n(F)$ -equivariant map  $\pi_{U_n} \twoheadrightarrow \tilde{\chi}$  for some character  $\tilde{\chi} : T_n(F) \rightarrow \mathbb{C}^\times$  such that  $\tilde{\chi}|_{T_n(\mathcal{O}_F)} = \chi$ . By Frobenius reciprocity,  $\pi$  is a subrepresentation of  $\text{n-Ind}_{B_n(F)}^{G(F)}(\tilde{\chi} \delta_{B_n}^{-1/2})$  (normalised induction, where  $\delta_{B_n}$  is the modulus character of  $B_n$ ). The latter representation is irreducible by our assumption that the  $\chi_i|_{k_F^\times}$  are distinct. (See Theorem 1.2.1 in [Kud94].) Thus  $\pi \cong \tilde{\chi}_1 |\cdot|^{(1-n)/2} \boxplus \cdots \boxplus \tilde{\chi}_n |\cdot|^{(n-1)/2}$ , so  $\text{rec}_{F,\mathbb{C}}(\pi) \cong (\tilde{\chi}_1 |\cdot|^{(1-n)/2} \oplus \cdots \oplus \tilde{\chi}_n |\cdot|^{(n-1)/2}) \circ \text{Art}_F^{-1}$  and  $N(\pi) = 0$  [HT01, p. 252]. As  $|\cdot|$  is unramified, this proves the proposition.  $\square$

### 3. $p$ -ADIC HODGE THEORY

In this section we develop the various results in  $p$ -adic Hodge theory that will be required in the sequel. In Subsections 3.1 and 3.2 we introduce basic definitions and results related respectively to strongly divisible modules and to Breuil modules; the novelty (to the extent that there is any) is that we allow coefficients and descent data. In fact, as noted in the introduction, we do not allow the most general form of either: rather we restrict to coefficients in the ring of integers  $\mathcal{O}_E$  of a finite extension  $E$  of  $\mathbb{Q}_p$  (or its residue field

$\mathbb{F}$  in the case of Breuil modules), and we restrict to descent data for tamely ramified extensions.

In Subsection 3.3 we pass to a detailed study of the particular Breuil modules that are relevant to our applications. This subsection culminates in Theorem 3.3.13, which is the key result that we will apply in the sequel.

### 3.1. Strongly divisible modules with coefficients and descent data.

In this subsection we extend certain results of [Sav05] on the categories of strongly divisible modules with coefficients and descent data to the case of lattices in potentially semistable Galois representations with Hodge–Tate weights in  $[0, p - 2]$  ([Sav05] works only with representations with Hodge–Tate weights in  $[0, 1]$ ). We do not attempt to work in the same level of generality as [Sav05], and in particular we only allow coefficients in the ring of integers of a finite extension of  $\mathbb{Q}_p$ .

We begin by recalling some results from Section 2.2 of [Sav05]. Let  $p$  be an odd prime and let  $E$  and  $K$  be finite extensions of  $\mathbb{Q}_p$  inside  $\overline{\mathbb{Q}_p}$ . Let  $K_0$  be the maximal unramified subfield of  $K$ , so that  $K_0 = W(k)[1/p]$ , where  $k$  is the residue field of  $K$ . Let  $\phi \in \text{Gal}(K_0/\mathbb{Q}_p)$  denote the arithmetic Frobenius. Let  $K/K'$  be a Galois extension, with  $K'$  a field lying between  $\mathbb{Q}_p$  and  $K$ . Fix the uniformiser  $p \in \mathbb{Q}_p$ , so we have a fixed embedding  $B_{\text{st}} \hookrightarrow B_{\text{dR}}$ .

*Definition 3.1.1.* A *filtered-* $(\varphi, N, K/K', E)$ -*module* of rank  $n$  is a free  $K_0 \otimes_{\mathbb{Q}_p} E$ -module  $D$  of rank  $n$  together with

- a  $\phi \otimes 1$ -semilinear automorphism  $\varphi$ ,
- a nilpotent  $K_0 \otimes_{\mathbb{Q}_p} E$ -linear endomorphism  $N$  such that  $N\varphi = p\varphi N$ ,
- a decreasing filtration  $(\text{Fil}^i D_K)_{i \in \mathbb{Z}}$  on  $D_K = K \otimes_{K_0} D$  which is exhaustive and separated, and
- a  $K_0$ -semilinear,  $E$ -linear action of  $\text{Gal}(K/K')$  which commutes with  $\varphi$  and  $N$  and preserves the filtration on  $D_K$ .

We say that such a module  $D$  is *weakly admissible* if the underlying filtered  $(\varphi, N, K/K, \mathbb{Q}_p)$ -module is weakly admissible.

Given a potentially semistable representation  $\rho : G_{K'} \rightarrow \text{GL}(V)$  on an  $n$ -dimensional  $E$ -vector space  $V$ , such that  $\rho|_{G_K}$  is semistable, we set

$$D_{\text{st}}^{K'}(V) = \text{Hom}_{G_K}(V, B_{\text{st}}),$$

a weakly admissible filtered  $(\varphi, N, K/K', E)$ -module of rank  $n$ . In the other direction, given a weakly admissible filtered  $(\varphi, N, K/K', E)$ -module  $D$ , we define

$$V_{\text{st}}^{K'}(D) = \text{Hom}_{\varphi, N}(D, B_{\text{st}}) \cap \text{Hom}_{\text{Fil}}(D_K, B_{\text{dR}}),$$

a finite-dimensional  $E$ -vector space with an action of  $G_{K'}$  via

$$(gf)(x) = g(f(\bar{g}^{-1}x))$$

where  $g \in G_{K'}$  and  $\bar{g}$  is the image of  $g$  in  $\text{Gal}(K/K')$ .

**Proposition 3.1.2.** *The functors  $D_{\text{st}}^{K'}$  and  $V_{\text{st}}^{K'}$  are mutually quasi-inverse, and provide equivalences of categories between the category of weakly admissible filtered  $(\varphi, N, K/K', E)$ -modules and the category of  $E$ -representations of  $G_{K'}$  which become semistable upon restriction to  $G_K$ .*

*Proof.* This follows from Proposition 2.9 of [Sav05].  $\square$

We now wish to consider  $\mathcal{O}_E$ -lattices inside potentially semistable Galois representations. This entails the consideration of strongly divisible modules. We begin with some basic definitions.

We suppose from now on that  $K/K'$  is a tamely ramified Galois extension with ramification index  $e(K/K')$ , and that we have fixed a uniformiser  $\pi \in K$  with  $\pi^{e(K/K')} \in K'$ . Write  $g(\pi) = h_g \pi$  for each  $g \in \text{Gal}(K/K')$ . Note that  $h_g \in W(k)$ . Let  $e$  be the absolute ramification index of  $K$ . Let  $E(u) \in W(k)[u]$  be the minimal polynomial of  $\pi$  over  $K_0$ . Let  $S$  be the  $p$ -adic completion of  $W(k)[u, \frac{u^{ie}}{i!}]_{i \in \mathbb{N}}$ . We let  $\varphi : S \rightarrow S$  denote the unique continuous,  $\phi$ -semilinear map with  $\varphi(u) = u^p$  and  $\varphi(u^{ie}/i!) = u^{iep}/i!$ , and we let  $N$  be the unique continuous,  $W(k)$ -linear derivation of  $S$  with  $N(u) = -u$  and  $N(u^{ie}/i!) = -ieu^{ie}/i!$ , so that  $N\varphi = p\varphi N$ . There is a descending filtration  $(\text{Fil}^i S)_{i \geq 0}$  on  $S$  given by letting  $\text{Fil}^i S$  be the  $p$ -adic completion of the ideal of  $S$  generated by  $E(u)^j/j!$  for  $j \geq i$  for each  $i \geq 0$ . For  $i \leq p-1$  we have  $\varphi(\text{Fil}^i S) \subset p^i S$ , and we let  $\varphi_i$  denote  $\varphi/p^i$  on  $\text{Fil}^i S$ . We let the group  $\text{Gal}(K/K')$  act on  $S$  by defining for each  $g \in \text{Gal}(K/K')$  the continuous ring isomorphism  $\hat{g} : S \rightarrow S$  with  $\hat{g}(w_i \frac{u^i}{[i/e]!}) = g(w_i) h_g^i \frac{u^i}{[i/e]!}$  (where  $w_i \in W(k)$ ). This action commutes with  $\varphi$  and  $N$  and preserves the filtration. (Note that  $\hat{g}(E(u)) = E(u)$  due to our assumption that  $\pi^{e(K/K')} \in K'$ .)

Let  $S_{\mathcal{O}_E} = S \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ , and extend the definitions of  $\text{Fil}$ ,  $\varphi$ ,  $\varphi_k$ ,  $N$ , and  $\hat{g}$  to  $S_{\mathcal{O}_E}$  in the obvious  $\mathcal{O}_E$ -linear fashion. Let  $S_E = S_{\mathcal{O}_E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ , and again extend the definitions of  $\text{Fil}$ ,  $\varphi$ ,  $N$ , and  $\hat{g}$  to  $S_E$  in the obvious fashion.

Let  $\mathcal{MF}(\varphi, N, K/K', E)$  be the category whose objects are finite free  $S_E$ -modules  $\mathcal{D}$  with

- a  $\varphi$ -semilinear,  $E$ -linear morphism  $\varphi_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$  such that the determinant of  $\varphi_{\mathcal{D}}$  with respect to some choice of  $S_{\mathbb{Q}_p}$  basis is invertible in  $S_{\mathbb{Q}_p}$  (this does not depend on the choice of basis);
- a decreasing filtration of  $\mathcal{D}$  by  $S_E$ -submodules  $\text{Fil}^i \mathcal{D}$ ,  $i \in \mathbb{Z}$ , with  $\text{Fil}^i \mathcal{D} = \mathcal{D}$  for  $i \ll 0$  and  $\text{Fil}^i S_E \text{Fil}^j \mathcal{D} \subset \text{Fil}^{i+j} \mathcal{D}$  for all  $j$  and all  $i \geq 0$ ;
- a  $K_0 \otimes_{\mathbb{Q}_p} E$ -linear map  $N : \mathcal{D} \rightarrow \mathcal{D}$  such that
  - for all  $s \in S_E$ ,  $x \in \mathcal{D}$  we have  $N(sx) = N(s)x + sN(x)$ ,
  - $N\varphi = p\varphi N$ ,
  - $N(\text{Fil}^i \mathcal{D}) \subset \text{Fil}^{i-1} \mathcal{D}$  for all  $i$ ;
- an  $S_E$ -semilinear action of  $\text{Gal}(K/K')$  on  $\mathcal{D}$  which commutes with  $\varphi$  and  $N$  and preserves each  $\text{Fil}^i \mathcal{D}$ .

Fix a positive integer  $r \leq p-2$ . Then the category  $\mathcal{O}_E\text{-Mod}_{\text{dd}}^r$  of strongly divisible  $\mathcal{O}_E$ -modules with descent data is defined to be the category of

finitely generated free  $S_{\mathcal{O}_E}$ -modules  $\widehat{\mathcal{M}}$  with a sub- $S_{\mathcal{O}_E}$ -module  $\text{Fil}^r \widehat{\mathcal{M}}$ , additive maps  $\varphi, N : \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}$ , and  $S_{\mathcal{O}_E}$ -semilinear bijections  $\hat{g} : \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}$  for each  $g \in \text{Gal}(K/K')$  such that the following conditions hold.

- $\text{Fil}^r \widehat{\mathcal{M}}$  contains  $(\text{Fil}^r S_{\mathcal{O}_E})\widehat{\mathcal{M}}$ ,
- $\text{Fil}^r \widehat{\mathcal{M}} \cap I\widehat{\mathcal{M}} = I \text{Fil}^r \widehat{\mathcal{M}}$  for all ideals  $I$  of  $\mathcal{O}_E$ ,
- $\varphi(sx) = \varphi(s)\varphi(x)$  for all  $s \in S_{\mathcal{O}_E}$ ,  $x \in \widehat{\mathcal{M}}$ ,
- $\varphi(\text{Fil}^r \widehat{\mathcal{M}})$  is contained in  $p^r \widehat{\mathcal{M}}$  and generates it over  $S_{\mathcal{O}_E}$ ,
- $N(sx) = N(s)x + sN(x)$  for all  $s \in S_{\mathcal{O}_E}$  and  $x \in \widehat{\mathcal{M}}$ ,
- $N\varphi = p\varphi N$ ,
- $E(u)N(\text{Fil}^r \widehat{\mathcal{M}}) \subset \text{Fil}^r \widehat{\mathcal{M}}$ ,
- for all  $g \in \text{Gal}(K/K')$ ,  $\hat{g}$  commutes with  $\varphi$  and  $N$ , and preserves  $\text{Fil}^r \widehat{\mathcal{M}}$ ,
- $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$  for all  $g_1, g_2 \in \text{Gal}(K/K')$ .

For any  $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-Mod}_{\text{dd}}^r$ , we define an  $\mathcal{O}_E$ -module  $\text{T}_{\text{st}}^{K'}(\widehat{\mathcal{M}})$  with an action of  $G_{K'}$  as follows: we let  $\widehat{\mathcal{A}}_{\text{st}}$  be the  $S$ -algebra with a filtration  $\text{Fil}^i \widehat{\mathcal{A}}_{\text{st}}$ , a Frobenius  $\varphi$ , and a monodromy operator  $N$  defined in Section 2.2.2 of [Bre99]. This ring has a natural action of  $G_{K'}$  (see the second paragraph of Section 3.4 of [Sav05] and the paragraph before Proposition 3.11 of *loc. cit.*), and the natural map  $\widehat{\mathcal{A}}_{\text{st}}[1/p] \rightarrow \text{B}_{\text{dR}}$  is  $G_{K'}$ -equivariant. [From Section 2.2.2 of [Bre99] we have  $\widehat{\mathcal{A}}_{\text{st}} = \{\sum_{n=0}^{\infty} a_n \frac{X^n}{n!} : a_n \in \mathbf{A}_{\text{cris}}, a_n \rightarrow 0\}$  and we define an action of  $G_{K'}$  on  $\widehat{\mathcal{A}}_{\text{st}}$  by setting  $g(\sum a_n \frac{X^n}{n!}) = \sum g(a_n) \frac{g(X)^n}{n!}$ , where

$$(3.1.3) \quad g(X) = \frac{g([\pi])}{[\pi]} \frac{\pi}{g(\pi)} (X + 1) - 1$$

and where  $[\pi]$  is the element of  $\mathbf{A}_{\text{cris}}$  corresponding to the Teichmüller representatives of a compatible system of  $p$ -th power roots of  $\pi$ . (Since  $g(\pi) = \pi$  for  $\pi \in G_K$ , this extends the natural action of  $G_K$  on  $\widehat{\mathcal{A}}_{\text{st}}$ .) From the second paragraph of Section 3.4 of [Sav05] and the paragraph before Proposition 3.11 of *loc. cit.* we see that this is the unique  $G_{K'}$ -action on  $\widehat{\mathcal{A}}_{\text{st}}$  such that the natural map  $\widehat{\mathcal{A}}_{\text{cris}} \hookrightarrow \widehat{\mathcal{A}}_{\text{st}}$  is  $G_{K'}$ -equivariant. The second paragraph of Section 3.4 of [Sav05] shows that the map  $f_{\pi} : \mathbf{A}_{\text{cris}} \rightarrow \text{B}_{\text{dR}}^+$  is  $G_{K'}$ -equivariant, and as explained in Section 2.2.2 of [Bre99] the map  $\widehat{\mathcal{A}}_{\text{st}}[1/p] \rightarrow \text{B}_{\text{dR}}$  is determined by the restriction of  $f_{\pi}$  to  $\mathbf{A}_{\text{cris}}$  and the fact that it sends the element  $1 + X \in \widehat{\mathcal{A}}_{\text{st}}$  to  $[\pi]/\pi$ . It follows that the map  $\widehat{\mathcal{A}}_{\text{st}}[1/p] \rightarrow \text{B}_{\text{dR}}$  is  $G_{K'}$ -equivariant, as required.] Then we put

$$\text{T}_{\text{st}}^{K'}(\widehat{\mathcal{M}}) := \text{Hom}_{\text{Fil}^r, \varphi, N}(\widehat{\mathcal{M}}, \widehat{\mathcal{A}}_{\text{st}})$$

(that is, the homomorphisms of  $S$ -modules which preserve  $\text{Fil}^r$  and commute with  $\varphi$  and  $N$ ). This inherits an  $\mathcal{O}_E$ -module structure from the  $\mathcal{O}_E$ -module structure on  $\widehat{\mathcal{M}}$ , and  $G_{K'}$  acts on  $\text{T}_{\text{st}}^{K'}(\widehat{\mathcal{M}})$  by

$$(gf)(x) = g(f(\hat{g}^{-1}x))$$

where  $g \in G_{K'}$  and  $\bar{g}$  is the image of  $g$  in  $\text{Gal}(K/K')$ . Note that this action is well defined since the map  $S \rightarrow \widehat{\mathcal{A}}_{\text{st}}$  is  $G_{K'}$ -equivariant and since the  $G_{K'}$ -action on  $\widehat{\mathcal{A}}_{\text{st}}$  commutes with  $\varphi$ ,  $N$  and preserves  $\text{Fil}^i \widehat{\mathcal{A}}_{\text{st}}$  for all  $i$ . (The first is clear by the definition of the  $G_{K'}$ -actions, since the map sends  $u$  to  $[\pi](1+X)^{-1}$ . The second follows from the definitions of these actions in Section 2.2.2 of [Bre99], and the fact that the  $G_{K'}$ -action commutes with  $\varphi$  and preserves all  $\text{Fil}^i$  on  $\widehat{\mathcal{A}}_{\text{cris}}$  by the second paragraph of Section 3.4 of [Sav05].)

The main result of this subsection is the following.

**Proposition 3.1.4.** *The functor  $\mathbb{T}_{\text{st}}^{K'}$  provides an anti-equivalence of categories between the category  $\mathcal{O}_E\text{-Mod}_{\text{dd}}^r$  of strongly divisible  $\mathcal{O}_E$ -modules with descent data and the category of  $G_{K'}$ -stable  $\mathcal{O}_E$ -lattices in finite-dimensional  $E$ -representations of  $G_{K'}$  which become semistable over  $K$  with Hodge–Tate weights lying in  $[-r, 0]$ .*

When  $K = K'$  and  $E = \mathbb{Q}_p$  this is the main result of [Liu08].

*Proof.* We write  $\mathbb{Z}_p\text{-Mod}^r$  for  $\mathcal{O}_E\text{-Mod}_{\text{dd}}^r$  and  $\mathbb{T}_{\text{st}}$  for  $\mathbb{T}_{\text{st}}^{K'}$  in the case when  $K = K'$  and  $E = \mathbb{Q}_p$ .

Suppose that  $\widehat{\mathcal{M}}$  is an object of  $\mathcal{O}_E\text{-Mod}_{\text{dd}}^r$ . Let  $\mathcal{D} := \widehat{\mathcal{M}}[\frac{1}{p}]$ . We extend  $\varphi$  and  $N$  by linearity and we define a filtration on  $\mathcal{D}$  as follows. We let  $\text{Fil}^r \mathcal{D} := (\text{Fil}^r \widehat{\mathcal{M}})[\frac{1}{p}]$  and

$$\text{Fil}^i \mathcal{D} := \begin{cases} \mathcal{D} & \text{for } i \leq 0, \\ \{x \in \mathcal{D} : E(u)^{r-i}x \in \text{Fil}^r \mathcal{D}\} & \text{for } 0 \leq i \leq r, \\ \sum_{j=0}^{i-1} (\text{Fil}^{i-j} S_{\mathbb{Q}_p}) (\text{Fil}^j \mathcal{D}) & \text{for } i > r, \text{ by induction.} \end{cases}$$

It is not hard to check that  $\mathcal{D}$  is an object of  $\mathcal{MF}(\varphi, N, K/K', E)$ . (To verify the condition about  $\varphi$ , pick elements  $f_i \in \text{Fil}^r \mathcal{M}$  such that  $\varphi_r(f_i)$  form an  $S$ -basis of  $\mathcal{M}$ , and let  $e_i := \varphi_r(f_i)$ . Write  $f_i = \sum a_{ij} \varphi_r(f_j)$  and  $\varphi(e_j) = \sum b_{jl} e_l$  for some matrices  $A = (a_{ij})$ ,  $B = (b_{jl})$  with entries in  $S$ . We have  $p^r I = \varphi(A)B$ , so  $\det(B) \in S_{\mathbb{Q}_p}^\times$ .) By [Bre97, §6] and Corollary 2.1.4 in [Bre99] we can write  $\mathcal{D} = D \otimes_{K_0} S_{\mathbb{Q}_p}$  for some weakly admissible filtered  $(\varphi, N, K/K, \mathbb{Q}_p)$ -module  $D$ , and we have  $D \cong \mathcal{D} \otimes_{S_{\mathbb{Q}_p, s}} K_0$  (where  $s(h(u)) = h(0)$ ) and  $D_K \cong \mathcal{D} \otimes_{S_{\mathbb{Q}_p, s_\pi}} K$  (where  $s_\pi(h(u)) = h(\pi)$ ), which induce  $\varphi$ ,  $N$  on  $D$  and the filtration on  $D_K$ . It follows that  $D$  inherits an  $E$ -action and a semilinear  $\text{Gal}(K/K')$ -action and it is straightforward to verify that it becomes a weakly admissible filtered  $(\varphi, N, K/K', E)$ -module. Note that  $\text{Fil}^{r+1} D_K = 0$  since  $\text{Fil}^{r+1} \mathcal{D} \subset (\text{Fil}^1 S_{\mathbb{Q}_p}) \mathcal{D}$ .

We associate an  $E$ -representation  $\mathbb{V}_{\text{st}}^{K'}(\mathcal{D})$  of  $G_{K'}$  to  $\mathcal{D}$  by defining

$$\mathbb{V}_{\text{st}}^{K'}(\mathcal{D}) := \text{Hom}_{\text{Fil}^r, \varphi, N}(\mathcal{D}, \widehat{\mathcal{A}}_{\text{st}}[1/p])$$

(that is, the homomorphisms of  $S$ -modules which preserve  $\text{Fil}^r$  and commute with  $\varphi$  and  $N$ ). This inherits an  $E$ -module structure from the  $E$ -module

structure on  $\mathcal{D}$ , and  $G_{K'}$  acts on  $V_{\text{st}}^{K'}(\mathcal{D})$  by

$$(gf)(x) = g(f(\hat{g}^{-1}x)),$$

where  $g \in G_{K'}$  and  $\bar{g}$  is the image of  $g$  in  $\text{Gal}(K/K')$ .

We have a natural  $\mathcal{O}_E[G_{K'}]$ -linear map  $T_{\text{st}}^{K'}(\widehat{\mathcal{M}}) \rightarrow V_{\text{st}}^{K'}(\mathcal{D})$ . We claim that there is a natural isomorphism of  $E[G_{K'}]$ -representations  $V_{\text{st}}^{K'}(\mathcal{D}) \rightarrow V_{\text{st}}^{K'}(D)$  given by  $f \mapsto f|_D$ . In the case that  $E = \mathbb{Q}_p$  and  $K' = K$ , this is established in the proof of Proposition 2.2.5 of [Bre02] (which also shows that  $f(D) \subset B_{\text{st}}^+ \hookrightarrow \widehat{A}_{\text{st}}[1/p]$ ), so in general this map gives a natural isomorphism  $V_{\text{st}}^{K'}(\mathcal{D}) \rightarrow V_{\text{st}}^{K'}(D)$  of the underlying  $\mathbb{Q}_p$ -vector spaces with an action of  $G_K$ . It is clear that this isomorphism is  $E$ -linear, and it remains to check that it is compatible with the action of  $G_{K'}$ . In the proof we just cited it is also shown that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f} & \widehat{A}_{\text{st}}[1/p] \\ f_\pi \downarrow & & \downarrow \\ D_K & \xrightarrow{f|_D \otimes 1} & B_{\text{st}} \otimes_{K_0} K \hookrightarrow B_{\text{dR}} \end{array}$$

Since the maps  $f_\pi, \widehat{A}_{\text{st}}[1/p] \rightarrow B_{\text{dR}}$ , and  $B_{\text{st}} \otimes_{K_0} K \rightarrow B_{\text{dR}}$  are  $G_{K'}$ -linear, so is  $V_{\text{st}}^{K'}(\mathcal{D}) \rightarrow V_{\text{st}}^{K'}(D)$ . By Proposition 2.2.5 of [Bre02] we know that  $T_{\text{st}}^{K'}(\widehat{\mathcal{M}}) \rightarrow V_{\text{st}}^{K'}(\mathcal{D})$  is a  $G_K$ -stable  $\mathbb{Z}_p$ -lattice. It follows that  $T_{\text{st}}^{K'}(\widehat{\mathcal{M}})$  is a  $G_{K'}$ -stable  $\mathcal{O}_E$ -lattice in  $V_{\text{st}}^{K'}(\mathcal{D})$ , and  $V_{\text{st}}^{K'}(\mathcal{D}) \cong V_{\text{st}}^{K'}(D)$  becomes semistable over  $K$  with Hodge–Tate weights in  $[-r, 0]$  by Proposition 3.1.2.

The faithfulness of  $T_{\text{st}}^{K'}$  follows immediately from the case when  $E = \mathbb{Q}_p$  and  $K' = K$ . Suppose now that  $\widehat{\mathcal{M}}_1, \widehat{\mathcal{M}}_2$  are objects of  $\mathcal{O}_E\text{-Mod}_{\text{dd}}^r$  and that we are given an  $E[G_{K'}]$ -linear map  $\theta : T_{\text{st}}^{K'}(\widehat{\mathcal{M}}_1) \rightarrow T_{\text{st}}^{K'}(\widehat{\mathcal{M}}_2)$ . From fullness in the case when  $E = \mathbb{Q}_p$  and  $K' = K$  we get a map  $\eta : \widehat{\mathcal{M}}_1 \rightarrow \widehat{\mathcal{M}}_2$  in  $\mathbb{Z}_p\text{-Mod}^r$  such that  $T_{\text{st}}(\eta) = \theta$ . We need to show that  $\eta$  is  $\mathcal{O}_E$ -linear and compatible with the  $\text{Gal}(K/K')$ -actions. If  $\lambda \in \mathcal{O}_E$ , then  $T_{\text{st}}(\lambda\eta) = T_{\text{st}}(\eta)\lambda = \lambda T_{\text{st}}(\eta) = T_{\text{st}}(\eta\lambda)$ , so  $\lambda\eta = \eta\lambda$ , as required. If  $g \in G_{K'}$ , then  $\hat{g}^{-1}\eta\hat{g}$  is a map in  $\mathbb{Z}_p\text{-Mod}^r$  and it is straightforward to show that  $T_{\text{st}}(\eta)g = gT_{\text{st}}(\eta)$  is equivalent to  $T_{\text{st}}(\hat{g}^{-1}\eta\hat{g}) = T_{\text{st}}(\eta)$ , so  $\hat{g}^{-1}\eta\hat{g} = \eta$ , as required.

It remains to check that  $T_{\text{st}}^{K'}$  is essentially surjective. Suppose that  $V$  is a finite-dimensional  $E$ -representation of  $G_{K'}$ , which becomes semistable over  $K$  with Hodge–Tate weights in  $[-r, 0]$ , and suppose that  $V_{\mathcal{O}_E}$  is a  $G_{K'}$ -stable  $\mathcal{O}_E$ -lattice in  $V$ . By Proposition 3.1.2 there is a weakly admissible filtered  $(\varphi, N, K/K', E)$ -module  $D$  with  $\text{Fil}^0 D_K = D_K$ ,  $\text{Fil}^{r+1} D_K = 0$ , and  $V_{\text{st}}^{K'}(D) \cong V$  as  $E$ -representations of  $G_{K'}$ .

Let  $\mathcal{D} := S_E \otimes_E D$ , a free  $S_E$ -module of finite rank. We endow  $\mathcal{D}$  with the structure of an object of  $\mathcal{MF}(\varphi, N, K/K', E)$  as follows:

- $\varphi_{\mathcal{D}}$  is given by the tensor product of  $\varphi$  on  $S_E$  and  $\varphi$  on  $D$ .
- $N := N \otimes \text{Id} + \text{Id} \otimes N : \mathcal{D} \rightarrow \mathcal{D}$ .

- $\text{Fil}^i \mathcal{D} := \mathcal{D}$  for  $i \leq 0$ , and by induction for  $i \geq 0$   
 $\text{Fil}^{i+1} \mathcal{D} := \{x \in \mathcal{D} \mid N(x) \in \text{Fil}^i \mathcal{D} \text{ and } f_\pi(x) \in \text{Fil}^{i+1} D_K\}$   
 where  $f_\pi : \mathcal{D} \rightarrow D_K$  is defined by  $\lambda \otimes x \mapsto s(\lambda)x$ , where  $s : S_E \rightarrow K \otimes_{\mathbb{Q}_p} E$  is the unique continuous  $K_0 \otimes_{\mathbb{Q}_p} E$ -linear map sending  $u^{ie}/i!$  to  $\pi^{ie}/i!$ .
- The action of  $\text{Gal}(K/K')$  is given by the tensor products of the actions on  $S_E$  and  $D$ .

By Theorem 2.3.5 of [Liu08], there is a strongly divisible lattice  $\widehat{\mathcal{M}} \subset \mathcal{D}$  of weight  $r$  with  $T_{\text{st}}(\widehat{\mathcal{M}})$  corresponding to  $V_{\mathcal{O}_E}$  under the isomorphism  $V_{\text{st}}^{K'}(\mathcal{D}) \cong V_{\text{st}}^{K'}(D) \cong V$ . Note that  $\widehat{\mathcal{M}} \in \mathbb{Z}_p\text{-Mod}^r$ . We now show that  $\widehat{\mathcal{M}}$  is stable under the actions of  $\mathcal{O}_E$  and  $\text{Gal}(K/K')$  on  $\mathcal{D}$ . Since  $\mathcal{O}_E$  is generated by  $\mathcal{O}_E^\times$  as a  $\mathbb{Z}_p$ -module, to check that  $\widehat{\mathcal{M}}$  is  $\mathcal{O}_E$ -stable it is enough to check that  $x\widehat{\mathcal{M}} = \widehat{\mathcal{M}}$  for all  $x \in \mathcal{O}_E^\times$ . Certainly  $x^{-1}\widehat{\mathcal{M}}$  is a strongly divisible lattice in  $\mathcal{D}$ . We also have

$$T_{\text{st}}(\widehat{\mathcal{M}}) = xT_{\text{st}}(\widehat{\mathcal{M}}) = T_{\text{st}}(x^{-1}\widehat{\mathcal{M}}),$$

so it suffices to show that if  $\widehat{\mathcal{M}}$  and  $\widehat{\mathcal{M}}'$  are strongly divisible lattices in  $\mathcal{D}$  with  $T_{\text{st}}(\widehat{\mathcal{M}}) = T_{\text{st}}(\widehat{\mathcal{M}}')$ , then  $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}'$ . But this is a formal consequence of Theorem 2.3.5 of [Liu08]. The proof that  $\widehat{\mathcal{M}}$  is  $\text{Gal}(K/K')$ -stable is very similar: if  $g \in G_{K'}$  then  $\hat{g}(\widehat{\mathcal{M}})$  is also a strongly divisible lattice in  $\mathcal{D}$ , and

$$T_{\text{st}}(\widehat{\mathcal{M}}) = gT_{\text{st}}(\widehat{\mathcal{M}}) = T_{\text{st}}(\hat{g}(\widehat{\mathcal{M}})),$$

so  $\hat{g}(\widehat{\mathcal{M}}) = \widehat{\mathcal{M}}$ , as required.

It remains to check that if  $I$  is an ideal of  $\mathcal{O}_E$  then  $I \text{Fil}^r \widehat{\mathcal{M}} = \text{Fil}^r \widehat{\mathcal{M}} \cap I\widehat{\mathcal{M}}$ , and that  $\widehat{\mathcal{M}}$  is free as an  $S_{\mathcal{O}_E}$ -module. The first follows since  $\text{Fil}^r \widehat{\mathcal{M}} = \widehat{\mathcal{M}} \cap \text{Fil}^r \mathcal{D}$ , by definition. The second follows as in the proof of Proposition 3.2.3.2 of [BM02]: let  $I$  denote the kernel of the ring homomorphism  $S \rightarrow W(k)$ ,  $h(u) \mapsto h(0)$ . Then  $\widehat{\mathcal{M}}/I\widehat{\mathcal{M}} \hookrightarrow D$  is a finitely generated  $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ -module which is a  $\mathbb{Z}_p$ -lattice. Since  $D$  is a finite free  $K_0 \otimes_{\mathbb{Q}_p} E$ -module and  $K_0/\mathbb{Q}_p$  is unramified one can check that  $\widehat{\mathcal{M}}/I\widehat{\mathcal{M}}$  is a finite free  $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ -module. (The point is that  $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$  is a product of discrete valuation rings.) As in *loc. cit.* we see that any lift of a basis to  $\widehat{\mathcal{M}}$  is a basis of  $\widehat{\mathcal{M}}$  as  $S_{\mathcal{O}_E}$ -module.  $\square$

**3.2. Breuil modules with descent data: part one.** As its title indicated, in this subsection we introduce Breuil modules with descent data, which morally speaking are the reductions mod  $p$  of strongly divisible modules with descent data, and which are the basic tool for studying the reductions mod  $p$  of lattices in potentially semistable Galois representations.

In fact we restrict ourselves to the case of representation which become semistable over a tamely ramified extension. Thus we carry over the running hypotheses of the preceding subsection, namely we consider a tamely ramified Galois extension  $K/K'$ , with  $K$  finite over  $\mathbb{Q}_p$ , and with ramification

index  $e(K/K')$ , and we furthermore assume given a uniformiser  $\pi \in K$  such that  $\pi^{e(K/K')} \in K'$ . We let  $e$  denote the absolute ramification index of  $K$ .

Let  $\mathbb{F}$  be a finite extension of  $\mathbb{F}_p$ . Recall that the category  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  of Breuil modules of weight  $r$  with descent data from  $K$  to  $K'$  and coefficients in  $\mathbb{F}$  consists of quintuples  $(\mathcal{M}, \mathcal{M}_r, \varphi_r, \hat{g}, N)$  where:

- $\mathcal{M}$  is a finitely generated  $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$ -module, free over  $k[u]/u^{ep}$ .
- $\mathcal{M}_r$  is a  $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$ -submodule of  $\mathcal{M}$  containing  $u^r \mathcal{M}$ .
- $\varphi_r : \mathcal{M}_r \rightarrow \mathcal{M}$  is  $\mathbb{F}$ -linear and  $\varphi$ -semilinear (where  $\varphi : k[u]/u^{ep} \rightarrow k[u]/u^{ep}$  is the  $p$ -th power map) with image generating  $\mathcal{M}$  as a  $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$ -module.
- $N : \mathcal{M} \rightarrow \mathcal{M}$  is  $k \otimes_{\mathbb{F}_p} \mathbb{F}$ -linear and satisfies  $N(ux) = uN(x) - ux$  for all  $x \in \mathcal{M}$ ,  $u^e N(\mathcal{M}_r) \subset \mathcal{M}_r$ , and  $\varphi_r(u^e N(x)) = cN(\varphi_r(x))$  for all  $x \in \mathcal{M}_r$ . Here,  $c \in (k[u]/u^{ep})^\times$  is the image of  $\varphi_1(E(u))$  under the natural map  $S \rightarrow k[u]/u^{ep}$ .
- $\hat{g} : \mathcal{M} \rightarrow \mathcal{M}$  are additive bijections for each  $g \in \text{Gal}(K/K')$ , preserving  $\mathcal{M}_r$ , commuting with the  $\varphi_r$ - and  $N$ -actions, and satisfying  $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$  for all  $g_1, g_2 \in \text{Gal}(K/K')$ . Furthermore, if  $a \in k \otimes_{\mathbb{F}_p} \mathbb{F}$ ,  $m \in \mathcal{M}$  then  $\hat{g}(aum) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i \hat{g}(m)$ .

*Remark 3.2.1.* It follows from the assumption that  $\mathcal{M}$  is a finitely generated  $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$ -module, which is free over  $k[u]/u^{ep}$ , that  $\mathcal{M}$  is in fact a free  $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$ -module. (The proof of Proposition 1.2 of [Sav08] goes through, with  $\mathcal{M}_r, \varphi_r$  replacing  $\mathcal{M}_1, \varphi_1$ .)

Suppose that  $\mathbb{F}$  is the residue field of a finite extension  $E$  of  $\mathbb{Q}_p$ . If  $\widehat{\mathcal{M}}$  is an object of  $\mathcal{O}_E\text{-Mod}_{\text{dd}}^r$ , then  $\mathcal{M} := (\widehat{\mathcal{M}}/\mathfrak{m}_E \widehat{\mathcal{M}}) \otimes_{S_{\mathcal{O}_E}/\mathfrak{m}_E S_{\mathcal{O}_E}} (k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$  is naturally an object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ , where we define  $\mathcal{M}_r$  to be the image of  $\text{Fil}^r \widehat{\mathcal{M}}$  in  $\mathcal{M}$ , the map  $\varphi_r$  is induced by  $(1/p^r)\varphi|_{\text{Fil}^r \widehat{\mathcal{M}}}$ , and  $N$  and  $\hat{g}$  are those coming from  $\widehat{\mathcal{M}}$ . (To see that  $\varphi_r$  is well defined on  $\mathcal{M}$  comes down to the fact that  $\ker(S/pS \rightarrow k[u]/u^{ep}) = \text{Fil}^p(S/pS)$ .)

We define a functor  $T_{\text{st}}$  from the category  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  to the category of finite-dimensional  $\mathbb{F}$ -representations of  $G_{K'}$  as follows: we let  $\widehat{\mathbf{A}}$  denote the  $k[u]/u^{ep}$ -algebra defined by the pushout square

$$\begin{array}{ccc} S & \longrightarrow & \widehat{\mathbf{A}}_{\text{st}} \\ \downarrow & & \downarrow \\ k[u]/u^{ep} & \longrightarrow & \widehat{\mathbf{A}} \end{array}$$

The vertical arrows induce a filtration  $\text{Fil}^r$ , maps  $\varphi_r$  and  $N$ , and an action of  $G_{K'}$  on the bottom row. (See §2.3 of [Car06] or §2.1 of [Car09].) Explicitly, for  $g \in G_{K'}$  we have  $g(u) = \frac{g(\pi)}{\pi}u$  and  $g(X) = \frac{g(\pi_1)}{\pi_1} \frac{\pi}{g(\pi)}(1+X) - 1$ , where  $\pi_1 \in \mathcal{O}_{\overline{K}}$  denotes the  $p$ -th root of  $\pi$  used in (3.1.3). (Note that this differs from the normalisation used in §4.2 in [Car09].) For  $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\text{dd}}^r$  we

define

$$\mathrm{T}_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{\mathrm{BrMod}}(\mathcal{M}, \widehat{\mathbb{A}}),$$

(that is, the  $k[u]/u^{ep}$ -homomorphisms which are compatible with the filtration,  $\varphi_r$  and  $N$ ). We let  $G_{K'}$  act by

$$(gf)(x) = gf(\widehat{g}^{-1}x).$$

**Lemma 3.2.2.** *The functor  $\mathrm{T}_{\mathrm{st}}$  is faithful. For any  $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$  we have*

$$\dim_{\mathbb{F}} \mathrm{T}_{\mathrm{st}}(\mathcal{M}) = \mathrm{rank}_{(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}} \mathcal{M}.$$

If  $\widehat{\mathcal{M}} \in \mathcal{O}_E\text{-Mod}_{\mathrm{dd}}^r$  and  $\mathcal{M} := (\widehat{\mathcal{M}}/\mathfrak{m}_E \widehat{\mathcal{M}}) \otimes_{S_{\mathcal{O}_E}/\mathfrak{m}_E S_{\mathcal{O}_E}} (k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$  denotes the Breuil module corresponding to the reduction of  $\widehat{\mathcal{M}}$ , then

$$\mathrm{T}_{\mathrm{st}}^{K'}(\widehat{\mathcal{M}}) \otimes_{\mathcal{O}_E} \mathbb{F} \cong \mathrm{T}_{\mathrm{st}}(\mathcal{M}).$$

*Proof.* The first two parts follow immediately from the corresponding results without coefficients and descent data (use Lemme 2.3.4 in [Car06] together with Corollary 2.3.3 of [CL09], respectively Lemme 2.3.1.2 in [Bre99]).

For the last part, we first define  $\varphi_r : \mathrm{Fil}^r \widehat{\mathcal{M}} \rightarrow \mathcal{M}$  by  $(1/p^r)\varphi$ . Then for any ideal  $I \subset \mathcal{O}_E$ , the  $S/I$ -module  $\widehat{\mathcal{M}}/I$  inherits a filtration (the image of  $\mathrm{Fil}^r \widehat{\mathcal{M}}$ ) and maps  $\varphi_r$ ,  $N$ ,  $\widehat{g}$  such that the natural map  $\widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}/I$  is compatible with all structures. Similarly,  $\widehat{\mathbb{A}}_{\mathrm{st}}/p^n$  inherits from  $\widehat{\mathbb{A}}_{\mathrm{st}}$  a filtration, maps  $\varphi_r$ ,  $N$ , and a  $G_{K'}$ -action. The natural map  $\mathrm{T}_{\mathrm{st}}^{K'}(\widehat{\mathcal{M}}) \rightarrow \varprojlim \mathrm{Hom}(\widehat{\mathcal{M}}/p^n, \widehat{\mathbb{A}}_{\mathrm{st}}/p^n)$  is an isomorphism, as explained just after Définition 4.1.1.1 in [Bre98]. (Here and in the following, we consider  $S$ -linear homomorphisms that respect filtration,  $\varphi_r$ ,  $N$ .) We consider the following three  $\mathcal{O}_E[G_{K'}]$ -linear maps:

$$(3.2.3) \quad \mathrm{T}_{\mathrm{st}}^{K'}(\widehat{\mathcal{M}})/p \rightarrow \mathrm{Hom}(\widehat{\mathcal{M}}/p, \widehat{\mathbb{A}}_{\mathrm{st}}/p)$$

$$(3.2.4) \quad \mathrm{Hom}(\widehat{\mathcal{M}}/p, \widehat{\mathbb{A}}_{\mathrm{st}}/p)/\mathfrak{m}_E \rightarrow \mathrm{Hom}(\widehat{\mathcal{M}}/\mathfrak{m}_E, \widehat{\mathbb{A}}_{\mathrm{st}}/p)$$

$$(3.2.5) \quad \mathrm{Hom}(\widehat{\mathcal{M}}/\mathfrak{m}_E, \widehat{\mathbb{A}}_{\mathrm{st}}/p) \rightarrow \mathrm{Hom}(\mathcal{M}, \widehat{\mathbb{A}}) = \mathrm{T}_{\mathrm{st}}(\mathcal{M})$$

The first two are isomorphisms by the argument used in the proof of Proposition 2.3.2.4 in [Bre99]. The last is an isomorphism by Lemme 2.3.4 of [Car06].  $\square$

**Proposition 3.2.6.** *Suppose that  $\mathcal{N} \in \mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$ , and that  $T'$  is a  $G_{K'}$ -subrepresentation of  $\mathrm{T}_{\mathrm{st}}(\mathcal{N})$  (so that in particular  $T'$  has the structure of an  $\mathbb{F}$ -vector space). Then there is a unique quotient  $\mathcal{N}'$  of  $\mathcal{N}$  such that if  $f : \mathcal{N} \rightarrow \mathcal{N}'$  is the quotient map, then  $\mathrm{T}_{\mathrm{st}}(f)$  is identified with the inclusion  $T' \hookrightarrow \mathrm{T}_{\mathrm{st}}(\mathcal{N})$ .*

*Proof.* We write  $\mathbb{F}_p\text{-BrMod}^r$  for the category  $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$  in case  $K' = K$  and  $\mathbb{F} = \mathbb{F}_p$ . In this case the proposition is proved in Proposition 2.2.5 of [Car09]. Thus we see that in general there is a unique quotient  $\mathcal{N}'$  of  $\mathcal{N}$  in  $\mathbb{F}_p\text{-BrMod}^r$  such that if  $f : \mathcal{N} \rightarrow \mathcal{N}'$  is the quotient map, then  $\mathrm{T}_{\mathrm{st}}(f)$  is the inclusion  $T' \hookrightarrow \mathrm{T}_{\mathrm{st}}(\mathcal{N})$ .

It remains to verify that the  $\text{Gal}(K/K')$ - and  $\mathbb{F}$ -actions descend to  $\mathcal{N}'$  via  $f$ . (It is then automatic that  $\mathcal{N}'$  becomes an object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  and the result follows.) Suppose for the moment that  $\mathcal{M} \in \mathbb{F}_p\text{-BrMod}^r$  and that  $g \in \text{Gal}(K/K')$ . We define the  $g$ -twist of  $\mathcal{M}$  as follows: we let  $\mathcal{M}^{(g)} := \mathcal{M} \otimes_{k[u]/u^{ep}, g} k[u]/u^{ep}$ . Then we have a  $g$ -semilinear bijection  $i : \mathcal{M} \rightarrow \mathcal{M}^{(g)}$  sending  $x$  to  $x \otimes 1$ . There is a unique way to make  $\mathcal{M}^{(g)}$  into an object of  $\mathbb{F}_p\text{-BrMod}^r$  such that the map  $i$  induces a bijection on filtrations and commutes with  $\varphi_r$  and  $N$ . (Note that  $c$  is fixed by  $\text{Gal}(K/K')$ .) It follows that for our  $\mathcal{N} \in \mathbb{F}\text{-BrMod}_{\text{dd}}^r$  the  $g$ -semilinear bijection  $\hat{g} : \mathcal{N} \rightarrow \mathcal{N}$  induces an isomorphism  $\hat{g} : \mathcal{N}^{(g)} \xrightarrow{\sim} \mathcal{N}$  in  $\mathbb{F}_p\text{-BrMod}^r$ . The assumption that  $T'$  be  $G_{K'}$ -stable implies that there is a unique  $\mathbb{F}_p[G_K]$ -linear isomorphism  $\theta(g) : \text{T}_{\text{st}}(\mathcal{N}') \xrightarrow{\sim} \text{T}_{\text{st}}(\mathcal{N}'^{(g)})$  such that  $\text{T}_{\text{st}}(\hat{g}) \circ \text{T}_{\text{st}}(f) = \text{T}_{\text{st}}(f^{(g)}) \circ \theta(g)$ . By the uniqueness of  $\mathcal{N}'$  we get a unique isomorphism  $\hat{g}_{\mathcal{N}'} : \mathcal{N}'^{(g)} \xrightarrow{\sim} \mathcal{N}'$  in  $\mathbb{F}_p\text{-BrMod}^r$  such that  $f \circ \hat{g} = \hat{g}_{\mathcal{N}'} \circ f$ .

Finally, a similar but easier argument shows that  $\mathcal{N}'$  inherits a natural action of  $\mathbb{F}^\times$  and thus (by  $\mathbb{F}_p$ -linearity) of  $\mathbb{F}$ , as required.  $\square$

*Remark 3.2.7.* As already remarked in the proof,  $\mathcal{N}$  is actually a quotient in the naive sense that  $\mathcal{N}' = \mathcal{N}/\mathcal{N}''$  for some  $\mathcal{N}'' \in \mathbb{F}\text{-BrMod}_{\text{dd}}^r$  (this follows easily from the proof of Proposition 2.2.5 of [Car09] and the exactness of the functor  $M_{\mathfrak{S}}$ , for which see Theorem 2.1.2 of [Car09]; we thank Xavier Caruso for explaining this to us).

It will be convenient for us to be able to employ duality in the calculations that follow. To this end, we briefly recall the duality theory for  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  developed in chapter V of [Car05] (see also Section 2.1 of [Car09]).

*Definition 3.2.8.* If  $\mathcal{M}$  is an object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ , then there is an object  $\mathcal{M}^*$  of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  such that

$$\text{T}_{\text{st}}(\mathcal{M}^*) = \text{T}_{\text{st}}(\mathcal{M})^\vee(r),$$

(i.e.  $\text{T}_{\text{st}}(\mathcal{M}^*)$  is the  $r$ -th Tate twist of the contragredient of  $\text{T}_{\text{st}}(\mathcal{M})$ ). Explicitly,  $\mathcal{M}^*$  is defined as follows.

- (i)  $\mathcal{M}^* := \text{Hom}_{k[u]/u^{ep}\text{-Mod}}(\mathcal{M}, k[u]/u^{ep})$ , with  $\mathbb{F}$ -structure inherited from that of  $\mathcal{M}$ .
- (ii)  $\mathcal{M}_r^* := \{f \in \mathcal{M}_r^* \mid f(\mathcal{M}_r) \in u^{er}k[u]/u^{ep}\}$ .
- (iii) Let  $\varphi_r : u^{er}k[u]/u^{ep} \rightarrow k[u]/u^{ep}$  be the unique semilinear map sending  $u^{er}$  to  $c^r$ . Then we define  $\varphi_r(f)$  by  $\varphi_r(f)(\varphi_r(x)) = \varphi_r(f(x))$  for all  $x \in \mathcal{M}_r$ ,  $f \in \mathcal{M}_r^*$ .
- (iv)  $N(f) := N \circ f - f \circ N$ , where  $N : k[u]/u^{ep} \rightarrow k[u]/u^{ep}$  is the unique  $k$ -linear derivation such that  $N(u) = -u$ .
- (v) Let  $g \in \text{Gal}(K/K')$  act on  $\mathcal{M}^*$  by  $(\hat{g}f)(x) = g(f(\hat{g}^{-1}x))$ , where  $\text{Gal}(K/K')$  acts on  $k[u]/u^{ep}$  by  $g(au^i) = g(a)(g(\pi)/\pi)^i u^i$  for  $a \in k$ .

(A priori we have a perfect,  $\mathbb{F}_p$ -bilinear pairing  $\langle \cdot, \cdot \rangle : \text{T}_{\text{st}}(\mathcal{M}) \times \text{T}_{\text{st}}(\mathcal{M}^*) \rightarrow \mathbb{F}_p(r)$  that is  $G_K$ -equivariant. Explicitly,  $\langle f, f' \rangle = \sum f(e_i)f'(e_i^*)$ , where  $(e_i)$

is any  $k[u]/u^{ep}$ -basis of  $\mathcal{M}$  and  $(e_i^*)$  is the dual basis of  $\mathcal{M}^*$ , by the proof of Theorem V.4.3.1 of [Car05]. It follows that  $\langle \cdot, \cdot \rangle$  is  $G_{K'}$ -equivariant and that  $\langle \lambda f, f' \rangle = \langle f, \lambda f' \rangle$  for all  $\lambda \in \mathbb{F}$ .) We have  $\mathcal{M}^{**} \cong \mathcal{M}$ . We define the covariant functor  $T_{\text{st}}^{*,r}$  from  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  to the category of  $\mathbb{F}$ -representations of  $G_{K'}$  by

$$T_{\text{st}}^{*,r}(\mathcal{M}) := T_{\text{st}}(\mathcal{M})^\vee(r),$$

so that

$$T_{\text{st}}^{*,r}(\mathcal{M}) = T_{\text{st}}(\mathcal{M}^*).$$

We then have the following immediate corollary of Proposition 3.2.6.

**Corollary 3.2.9.** *Suppose that  $\mathcal{M} \in \mathbb{F}\text{-BrMod}_{\text{dd}}^r$ , and that  $T'$  is a  $G_{K'}$ -subrepresentation of  $T_{\text{st}}^{*,r}(\mathcal{M})$  (so that in particular  $T'$  has the structure of an  $\mathbb{F}$ -vector space). Then there is a unique subobject  $\mathcal{M}'$  of  $\mathcal{M}$  such that if  $f : \mathcal{M}' \rightarrow \mathcal{M}$  is the inclusion map, then  $T_{\text{st}}^{*,r}(f)$  is identified with the inclusion  $T' \hookrightarrow T_{\text{st}}^{*,r}(\mathcal{M})$ .*

*Proof.* Let  $\mathcal{N} := \mathcal{M}^*$ , so that  $T'$  is a sub- $G_{K'}$ -representation of  $T_{\text{st}}(\mathcal{N}) = T_{\text{st}}^{*,r}(\mathcal{M})$ . Then by Proposition 3.2.6, there is a unique quotient  $\mathcal{N}'$  of  $\mathcal{N}$  such that applying  $T_{\text{st}}$  to the natural projection map  $\mathcal{N} \rightarrow \mathcal{N}'$  gives the inclusion  $T' \hookrightarrow T_{\text{st}}(\mathcal{N})$ . Then  $\mathcal{M}' := (\mathcal{N}')^*$  is the required subobject of  $\mathcal{M}$ , and uniqueness is clear.  $\square$

*Remark 3.2.10.* Just as in Remark 3.2.7,  $\mathcal{M}'$  is a subobject of  $\mathcal{M}$  in the naive sense that it is a sub- $(k[u]/u^{ep} \otimes \mathbb{F})$ -module of  $\mathcal{M}$ , which inherits the structure of an object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  from  $\mathcal{M}$  in the obvious way.

Finally, we recall something of the theory of maximal objects from [Car09] (see Sections 4.1 and 4.2 of [Car09] for a proof of the existence of maximal Breuil modules with descent data and coefficients). If  $\mathcal{M}$  is an object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ , the corresponding maximal object  $\text{Max}(\mathcal{M})$  is an object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  together with a morphism  $\iota_{\text{Max}}^{\mathcal{M}} : \mathcal{M} \rightarrow \text{Max}(\mathcal{M})$  such that  $T_{\text{st}}^{*,r}(\iota_{\text{Max}}^{\mathcal{M}})$  is an isomorphism. Furthermore, it satisfies the following universal property: if  $f : \mathcal{M} \rightarrow \mathcal{M}'$  is such that  $T_{\text{st}}^{*,r}(f)$  is an isomorphism, then there exists a unique morphism  $g : \mathcal{M}' \rightarrow \text{Max}(\mathcal{M})$  such that  $g \circ f = \iota_{\text{Max}}^{\mathcal{M}}$ .

**3.3. Breuil modules with descent data: part two.** We now specialise to the particular situation of interest to us in this paper. Let  $K_0$  be the unique unramified extension of  $\mathbb{Q}_p$  of degree  $d$ , and let  $K = K_0((-p)^{1/(p^d-1)})$ . Let  $K' = K_0$ . Fix  $\pi = (-p)^{1/(p^d-1)}$ . We write  $\tilde{\omega}_d : \text{Gal}(K/K_0) \rightarrow K_0^\times$  for the character  $g \mapsto g(\pi)/\pi$ , and we let  $\omega_d$  be the reduction of  $\tilde{\omega}_d$  modulo  $\pi$ . (By inflation we can also think of  $\tilde{\omega}_d$  and  $\omega_d$  as characters of  $I_{K_0} = I_{\mathbb{Q}_p}$ . Note that  $\omega_d$  is a tame fundamental character of niveau  $d$  and that  $\tilde{\omega}_d$  is the Teichmüller lift of  $\omega_d$ .)

Let  $E/\mathbb{Q}_p$  be a finite extension such that  $E$  contains the images of all embeddings  $K \hookrightarrow \overline{\mathbb{Q}_p}$ .

If  $\rho : G_{K_0} \rightarrow \mathrm{GL}_n(E)$  is a potentially semistable representation which becomes semistable over  $K$ , then the associated inertial type is a representation of  $I_{K_0}$  which becomes trivial when restricted to  $I_K$ , so we may think of it as a representation of  $\mathrm{Gal}(K/K_0) \cong I_{K_0}/I_K$ . Since this group is abelian, this representation is isomorphic to a direct sum  $\chi_1 \oplus \cdots \oplus \chi_n$  where each  $\chi_i : \mathrm{Gal}(K/K_0) \rightarrow \mathcal{O}_E^\times$  is a character.

Let  $\mathbb{F}$  be the residue field of  $\mathcal{O}_E$ . Let  $\phi$  be the arithmetic Frobenius on  $k$ , and let  $\sigma_0 : k \hookrightarrow \mathbb{F}$  be a fixed embedding. Inductively define  $\sigma_1, \dots, \sigma_{d-1}$  by  $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$ ; we will often consider the numbering to be cyclic, so that  $\sigma_d = \sigma_0$ . There are idempotents  $e_i \in k \otimes_{\mathbb{F}_p} \mathbb{F}$  such that if  $M$  is any  $k \otimes_{\mathbb{F}_p} \mathbb{F}$ -module, then  $M = \bigoplus_i e_i M$ , and  $e_i M$  is the subset of  $M$  consisting of elements  $m$  for which  $(x \otimes 1)m = (1 \otimes \sigma_i(x))m$  for all  $x \in k$ . Note that  $(\phi \otimes 1)(e_i) = e_{i+1}$  for all  $i$ .

**Proposition 3.3.1.** *Maintain our current assumptions on  $K_0$ ,  $K$  and  $K'$ . Suppose that  $\rho : G_{K_0} \rightarrow \mathrm{GL}_n(E)$  is a continuous representation such that  $\rho|_{G_K}$  is semistable with Hodge–Tate weights contained in  $[-r, 0]$ , with  $r \leq p - 2$ , and that  $\rho$  has inertial type  $\chi_1 \oplus \cdots \oplus \chi_n$  as above. Let  $\bar{\rho}$  be the reduction mod  $\mathfrak{m}_E$  of a  $G_{K_0}$ -stable  $\mathcal{O}_E$ -lattice in  $\rho$ . Then there is an element  $\mathcal{M}$  of  $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$  with*

$$\mathrm{T}_{\mathrm{st}}^{*,r}(\mathcal{M}) \cong \bar{\rho},$$

and  $\mathcal{M}$  has a  $(k[u]/u^{ep} \otimes_{\mathbb{F}_p} \mathbb{F})$ -basis  $v_1, \dots, v_n$  such that  $\hat{g}(v_i) = (1 \otimes \bar{\chi}_i(g))v_i$  for all  $g \in \mathrm{Gal}(K/K_0)$ .

*Proof.* By Proposition 3.1.4, there is an element  $\widehat{\mathcal{M}}$  of  $\mathcal{O}_E\text{-Mod}_{\mathrm{dd}}^r$  such that  $\mathrm{T}_{\mathrm{st}}^{K'}(\widehat{\mathcal{M}})$  is the given lattice in  $\rho$ . Then we let  $\bar{\mathcal{N}} := (\widehat{\mathcal{M}}/\mathfrak{m}_E \widehat{\mathcal{M}}) \otimes_{S_E/\mathfrak{m}_E S_E} k[u]/u^{ep}$ , and  $\mathcal{M} := \bar{\mathcal{N}}^*$ , so that  $\mathrm{T}_{\mathrm{st}}^{*,r}(\mathcal{M}) \cong \bar{\rho}$ . In the case  $r = 1$ , the claim about the form of the descent data is proved in Section 5 of [GS10b]; but the proof given there makes no use of the fact that  $r = 1$ , and extends immediately to this more general situation.  $\square$

**Lemma 3.3.2.** *Maintain our current assumptions on  $K_0$ ,  $K$  and  $K'$ , so  $e = p^d - 1$ . Then every rank one object of  $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$  may be written in the form:*

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}) \cdot m$ ,
- $e_i \mathcal{M}_r = u^{r_i} e_i \mathcal{M}$ ,
- $\varphi_r(\sum_{i=0}^{d-1} u^{r_i} e_i m) = \lambda m$  for some  $\lambda \in (k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ ,
- $\hat{g}(m) = (\sum_{i=0}^{d-1} (\omega_d(g)^{k_i} \otimes 1) e_i) m$  for all  $g \in \mathrm{Gal}(K/K_0)$ , and
- $N(m) = 0$ .

Here the integers  $0 \leq r_i \leq er$  and  $k_i$  satisfy  $k_i \equiv p(k_{i-1} + r_{i-1}) \pmod{e}$  for all  $i$ . Conversely, any module  $\mathcal{M}$  of this form is a rank one object of  $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^r$ . Furthermore,

$$\mathrm{T}_{\mathrm{st}}^{*,r}(\mathcal{M})|_{I_{K_0}} \cong \sigma_0 \circ \omega_d^{k_0},$$

where  $\kappa_0 \equiv k_0 + s_0 \pmod{e}$ , and

$$s_0 := p(r_0 p^{d-1} + r_1 p^{d-2} + \cdots + r_{d-1}) / (p^d - 1).$$

*Proof.* If  $\mathcal{M}$  is a rank one object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ , then it is of the claimed form by Theorem 3.5 of [Sav08], except for the statement that  $N(m) = 0$ . (Note that the cited proof only uses that  $u^e \mathcal{M} \subset \mathcal{M}_1$  at one point, once all the work is done: to know that  $r_i \leq e$  in the statement of the theorem. So we can just replace  $\mathcal{M}_1$  by  $\mathcal{M}_r$  throughout.) Now it is easy to see that  $N(\mathcal{M}_r) \subset \mathcal{M}_r$ . Then  $N(m) = 0$  by using that  $\varphi_r(u^e N(x)) = cN(\varphi_r(x))$  for all  $x \in \mathcal{M}_r$ .

Conversely it is straightforward to verify that any module  $\mathcal{M}$  of the above form is a rank one object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ . The calculation of  $T_{\text{st}}^{*,r}(\mathcal{M})|_{I_{K_0}}$  may be performed in the same way as in Example 3.7 of [Sav08] or Corollaire 4.4.2 of [Car09]; we sketch the details of an approach following Example 3.7 of [Sav08].

Given  $\mathcal{M}$  as in the statement of the lemma, define another element  $\mathcal{M}'$  of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  by

- $\mathcal{M}' = ((k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}) \cdot m'$ ,
- $e_i \mathcal{M}'_r = e_i \mathcal{M}'$ ,
- $\varphi_r(\sum_{i=0}^{d-1} e_i m') = \lambda m'$ ,
- $\hat{g}(m') = (\sum_{i=0}^{d-1} (\omega_d(g)^{p^i \kappa_0} \otimes 1) e_i) m'$  for all  $g \in \text{Gal}(K/K_0)$ , and
- $N(m') = 0$ .

We can define a morphism  $\mathcal{M}' \rightarrow \mathcal{M}$  by mapping  $e_i m' \mapsto u^{s_i} e_i m$ , where

$$s_i := p(r_i p^{d-1} + r_{i+1} p^{d-2} + \cdots + r_{i+d-1}) / (p^d - 1).$$

One easily checks that this is a morphism in the category  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ ; for example,  $s_i \geq r_i$ , so the map is compatible with the filtration, and since  $s_{i+1} = p(s_i - r_i)$  the morphism commutes with  $\varphi_r$ . Since  $T_{\text{st}}^{*,r}$  is faithful we have a nonzero map  $T_{\text{st}}^{*,r}(\mathcal{M}') \rightarrow T_{\text{st}}^{*,r}(\mathcal{M})$ , which must be an isomorphism because both source and target are one-dimensional  $\mathbb{F}$ -vector spaces.

The calculation of  $T_{\text{st}}^{*,r}(\mathcal{M}')$  may now be carried out in the same fashion as in Example 3.7 of [Sav08]. We sketch the argument. Denote a Teichmüller lift by a tilde. One checks without difficulty that  $\mathcal{M}'$  may be lifted to a strongly divisible module with descent data, contained inside the object of  $\mathcal{MF}(\varphi, N, K/K', W(\mathbb{F})[1/p])$  associated to the weakly admissible filtered  $(\varphi, N, K/K', W(\mathbb{F})[1/p])$ -module  $D = (K_0 \otimes_{\mathbb{Q}_p} W(\mathbb{F})[1/p]) \cdot v$ , where  $N = 0$ ,  $\varphi(v) = \tilde{\lambda} p^r v$ ,  $g(v) = (1 \otimes \tilde{\sigma}_0 \circ \tilde{\omega}_d(g)^{\kappa_0}) v$ , and  $\text{Fil}^i(K \otimes_{K'} D)$  is 0 for  $i \geq r+1$  and  $K \otimes_{K'} D$  for  $i \leq r$ . The corresponding Galois representation can then be computed as in Example 2.13 of [Sav05]: we find  $V_{\text{st}}^{K'}(D)^\vee(r)|_{I_{K_0}} \cong \tilde{\sigma}_0 \circ \tilde{\omega}_d^{\kappa_0}$ , which indeed reduces to  $\sigma_0 \circ \omega_d^{\kappa_0}$ .  $\square$

**Lemma 3.3.3.** *Maintain our assumptions on  $K_0$ ,  $K$  and  $K'$ , so  $e = p^d - 1$ . Suppose that  $\mathcal{M}$  is the rank one object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  of Lemma 3.3.2. Then there exists a basis  $\tilde{m}$  of  $\text{Max}(\mathcal{M})$  such that*

- $e_i \text{Max}(\mathcal{M})_r = u^{er} e_i \text{Max}(\mathcal{M})$ ,
- $\varphi_r(\sum_{i=0}^{d-1} u^{er} e_i \tilde{m}) = \lambda \tilde{m}$ ,
- $\hat{g}(\tilde{m}) = \sum_{i=0}^{d-1} (\omega_d(g)^{p^i s} \otimes 1) e_i \tilde{m}$  for all  $g \in \text{Gal}(K/K_0)$ , and
- $N(\tilde{m}) = 0$ ,

with  $s \equiv \kappa_0 - \frac{er}{p-1} \pmod{e}$ , where we keep the notation of Lemma 3.3.2.

In particular, the Breuil module in Lemma 3.3.2 is maximal iff  $r_i = er$  for all  $i$ .

*Proof.* Let  $\mathcal{M}'$  denote the Breuil module defined in the statement of the lemma. Let

$$t_i = \frac{erp}{p-1} - \frac{r_i p^d + r_{i+1} p^{d-1} + \cdots + r_{i+d-1} p}{e}.$$

It is an integer, and since  $0 \leq r_i \leq er$ , it lies in the interval  $[0, \frac{erp}{p-1}]$ . Define a map  $f : \mathcal{M} \rightarrow \mathcal{M}'$  by  $f(e_i m) := u^{t_i} e_i \tilde{m}$ . Then  $f \neq 0$  since  $r < p-1$ . To see that  $f$  commutes with  $\varphi_r$ , use that  $t_{i+1} = p(r_i + t_i - er)$ . Since  $p$  divides  $t_i$ , it follows that  $f$  commutes with  $N$ . It is straightforward to check that  $f$  commutes with descent data. Since  $\mathbb{T}_{\text{st}}^{*,r}$  is faithful, it follows that  $\mathbb{T}_{\text{st}}^{*,r}(f)$  is an isomorphism.

It remains to verify that  $\mathcal{M}'$  is maximal. If we have a map of Breuil modules  $h : \mathcal{M}' \rightarrow \mathcal{N}$  such that  $\mathbb{T}_{\text{st}}^{*,r}(h)$  is an isomorphism,  $\mathcal{N}$  has to be of rank one and we can find a basis  $n$  such that

- $e_i \mathcal{N}_r = u^{r'_i} e_i \mathcal{N}$ ,
- $\varphi_r(\sum_{i=0}^{d-1} u^{r'_i} e_i n) = \mu n$ ,
- $\hat{g}(n) = \sum_{i=0}^{d-1} (\omega_d(g)^{k'_i} \otimes 1) e_i n$  for all  $g \in \text{Gal}(K/K_0)$ , and
- $N(n) = 0$ ,

for some  $\mu \in (k \otimes \mathbb{F})^\times$  and integers  $r'_i \in [0, er]$  and  $k'_i$  such that  $k'_i \equiv p(k'_{i-1} + r'_{i-1})$  modulo  $e$ . We can write  $h(e_i \tilde{m}) = \nu_i (u^{m_i} + O(u^{m_i+1})) e_i n$  for some  $\nu_i \in \mathbb{F}^\times$  and  $m_i \geq 0$ . Since  $h$  commutes with  $\varphi_r$  we find that

$$(3.3.4) \quad m_{i+1} = p(m_i + er - r'_i),$$

unless both sides are at least  $ep$ . But  $h \neq 0$  implies that  $m_i < ep$  for some  $i$  and if  $m_i < C$  for some constant  $C \leq ep$ , then (3.3.4) shows that  $m_{i-1} < C/e$ . Inductively we thus find that  $m_i = 0$  for all  $i$ , so  $r'_i = er$  for all  $i$  and  $h$  is an isomorphism.  $\square$

*Remark 3.3.5.* Similarly (or by duality) we find that  $\mathcal{M}$  in Lemma 3.3.2 is minimal if and only if  $r_i = 0$  for all  $i$ .

The following lemma works more generally than for our particular choice of extension  $K/K'$  above, and so just for the duration of its statement and proof, we suppress our running definitions of  $K$  and  $K'$ , and return to the more general context of Subsection 3.2.

**Lemma 3.3.6.** *Suppose that  $K/K'$  is totally ramified and that  $\mathbb{F}$  contains an embedding of  $k$ . Suppose that  $\mathcal{M}$  is a rank  $d$  object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ . Then*

$\mathcal{M}$  has a basis  $m^{(j)}$  ( $1 \leq j \leq d$ ) such that for all  $j$  we have  $m^{(j)} \in \varphi_r(\mathcal{M}_r)$  and  $(k \otimes_{\mathbb{F}_p} \mathbb{F}) \cdot m^{(j)}$  is  $\text{Gal}(K/K')$ -stable.

*Proof.* We let  $d = [k : \mathbb{F}_p]$ . Since we are assuming that  $\mathbb{F}$  contains an embedding of  $k$ , we may define the idempotents  $e_i$  as above. To prove the lemma, it suffices to show that  $e_0\mathcal{M}$  has an  $\mathbb{F}[u]/u^{ep}$ -basis  $n^{(j)}$  such that  $\mathbb{F} \cdot n^{(j)}$  is  $\text{Gal}(K/K')$ -stable. Thus it is enough to show that there are generators  $\alpha^{(j)}$  ( $1 \leq j \leq d$ ) of  $e_{d-1}\mathcal{M}_r$  as an  $\mathbb{F}[u]/u^{ep}$ -module such that  $\mathbb{F} \cdot \alpha^{(j)}$  is  $\text{Gal}(K/K')$ -stable. (Note that  $(\varphi_r(\alpha^{(j)}))_{j=1}^d$  is then a basis of  $e_0\mathcal{M}$  as  $\mathbb{F}[u]/u^{ep}$ -module.)

Note that  $\text{Gal}(K/K')$  is abelian of order  $e(K/K')$ , which is prime to  $p$ . It acts trivially on  $k$ . Moreover  $k$  contains the  $e(K/K')$ -th roots of unity. Thus  $e_{d-1}\mathcal{M}_r$  decomposes as a direct sum of one-dimensional  $\mathbb{F}$ -subspaces that are  $\text{Gal}(K/K')$ -stable. Pick an  $\mathbb{F}$ -basis  $\beta_l$  of  $e_{d-1}\mathcal{M}_r$  such that  $\mathbb{F} \cdot \beta_l$  is  $\text{Gal}(K/K')$ -stable for all  $l$ . Pick a subset  $\alpha^{(j)}$  ( $1 \leq j \leq d$ ) of that basis that forms a basis inside  $e_{d-1}(\mathcal{M}_r/u\mathcal{M}_r) \cong \mathbb{F}^d$ . (The latter isomorphism follows from the theorem on elementary divisors, since  $r < p$ .) By Nakayama's lemma the  $\alpha^{(j)}$  generate  $e_{d-1}\mathcal{M}_r$ , as required.  $\square$

**Lemma 3.3.7.** *Suppose that  $M$  and  $M'$  are free  $\mathbb{F}[u]/u^{ep}$ -modules of rank  $d$  and that  $f : M \rightarrow M'$  is an  $\mathbb{F}[u]/u^{ep}$ -linear map. Suppose that  $M$  has basis  $(e_j)_{j=1}^d$ . If  $f(e_1) \in u^{n+1}M'$  for some positive integer  $n + 1 \leq ep$ , then  $\text{im}(f) \not\supset u^n M'$ .*

*Proof.* Suppose to the contrary that we have the containment  $\text{im}(f) \supset u^n M'$ . Let  $I$  denote the  $\mathbb{F}$ -vector space  $\text{im}(f)/u \text{im}(f)$ . On the one hand, the images of the  $f(e_j)$  span  $I$ , and by assumption  $f(e_1)$  is zero in  $I$ . Thus  $\dim_{\mathbb{F}} I < d$ . On the other hand, by the theorem on elementary divisors we can pick a basis  $(e'_j)_{j=1}^d$  of  $M'$  and non-negative integers  $a_j$  (possibly greater than or equal to  $ep$ ) such that the  $(u^{a_j} e'_j)_{j=1}^d$  span  $\text{im}(f)$  as an  $\mathbb{F}[u]/u^{ep}$ -module. Since  $u^n M'$  is spanned by the  $(u^n e'_j)_{j=1}^d$ , we see that  $a_j \leq n < ep$  for all  $j$ . Therefore  $\dim_{\mathbb{F}} I = d$  and we get a contradiction, as required.  $\square$

We owe the following argument to Xavier Caruso. We would like to thank him for letting us include it here. Without it, we would need a slightly stronger condition on  $(a, b, c)$  in Theorem 3.3.13 below. (Namely we would need to demand that  $a - b > 3$ ,  $b - c > 3$ ,  $a - c < p - 4$  and that none of  $2a - b - c$ ,  $2b - a - c$ ,  $2c - a - b$  are congruent to an element of  $[-3, 3]$  modulo  $p - 1$ .)

**Proposition 3.3.8.** *Maintain our assumptions on  $K_0$ ,  $K$  and  $K'$ , so  $e = p^d - 1$ . Assume that  $r < \frac{p-1}{2}$ . Suppose that  $\mathcal{M}$  is a rank  $d$  object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$ . Suppose that  $T_{\text{st}}^{*,r}(\mathcal{M})$  extends to an irreducible representation of  $G_{\mathbb{Q}_p}$ . Suppose that  $\mathcal{N}$  is a rank one object of  $\mathbb{F}\text{-BrMod}_{\text{dd}}^r$  and that we are given a map  $\mathcal{N} \rightarrow \mathcal{M}$  of Breuil modules that is injective (as an  $\mathbb{F}$ -linear map).*

Suppose that  $\mathcal{M}$  has a basis  $m^{(j)}$  ( $0 \leq j \leq d-1$ ) with  $\hat{g}(m^{(j)}) = (\omega_d(g)^{a_j} \otimes 1)m^{(j)}$  for all  $j$  and all  $g \in \text{Gal}(K/K_0)$ , where the integers  $a_j$  satisfy the condition that  $(a_j - a_l)/p$  is congruent to an element of the interval  $(\frac{er}{p-1}, \frac{e(p-1-r)}{p-1})$  modulo  $e$  whenever  $j \neq l$ . Then there exists a basis  $n$  of  $\mathcal{N}$  and a permutation  $\pi$  of  $\{0, 1, \dots, d-1\}$  such that

$$\hat{g}(n) = \sum_{i=0}^{d-1} (\omega_d(g)^{a_{\pi(i)}} \otimes 1) e_i n.$$

*Proof.* Since  $\mathcal{N} \rightarrow \mathcal{M}$  is nonzero and  $T_{\text{st}}^{*,r}$  is faithful we see that  $T_{\text{st}}^{*,r}(\mathcal{N})$  is a one-dimensional subrepresentation  $\mathbb{F}(\chi)$  of  $T_{\text{st}}^{*,r}(\mathcal{M})$ . As  $T_{\text{st}}^{*,r}(\mathcal{M})$  extends to an irreducible representation of  $G_{\mathbb{Q}_p}$  we have  $T_{\text{st}}^{*,r}(\mathcal{M}) \cong \bigoplus_{j=0}^{d-1} \mathbb{F}(\chi^{p^j})$  and the characters  $\chi^{p^j}$  ( $0 \leq j \leq d-1$ ) are pairwise distinct. By Lemma 3.2.9 there are rank one Breuil modules  $\mathcal{N}^{(j)}$  for  $0 \leq j \leq d-1$  together with maps  $\mathcal{N}^{(j)} \rightarrow \mathcal{M}$  that are injective as  $\mathbb{F}$ -linear maps and such that  $T_{\text{st}}^{*,r}(\mathcal{N}^{(j)}) \cong \mathbb{F}(\chi^{p^j})$ . We can take  $\mathcal{N}^{(0)} = \mathcal{N}$ . It follows that we have a map  $\alpha : \bigoplus_j \mathcal{N}^{(j)} \rightarrow \mathcal{M}$  such that  $T_{\text{st}}^{*,r}(\alpha)$  is an isomorphism. By the universal property of maximal objects and Lemma 1.5.1 of [Car09] we have a map  $\beta : \mathcal{M} \rightarrow \bigoplus_j \text{Max}(\mathcal{N}^{(j)})$  such that  $\beta \circ \alpha$  is a direct sum of the natural maps  $\mathcal{N}^{(j)} \rightarrow \text{Max}(\mathcal{N}^{(j)})$ .

By Lemma 3.3.2 there is a basis  $n^{(j)}$  of  $\mathcal{N}^{(j)}$ , integers  $r_i^{(j)} \in [0, er]$  and  $x_i^{(j)} \in \{a_1, \dots, a_d\}$ , and  $\lambda^{(j)} \in (k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$  such that

- $e_i \mathcal{N}_r^{(j)} = u^{r_i^{(j)}} e_i \mathcal{N}^{(j)}$ ,
- $\varphi_r(\sum_{i=0}^{d-1} u^{r_i^{(j)}} e_i n^{(j)}) = \lambda^{(j)} n^{(j)}$ ,
- $\hat{g}(n^{(j)}) = \sum_{i=0}^{d-1} (\omega_d(g)^{x_i^{(j)}} \otimes 1) e_i n^{(j)}$  for all  $g \in \text{Gal}(K/K_0)$ , and
- $N(n^{(j)}) = 0$ .

(For the descent data note that  $e_i(\mathcal{N}/u\mathcal{N})$  injects into  $e_i(\mathcal{M}/u\mathcal{M})$ .) By Lemma 3.3.3 we know that there is a basis  $\tilde{n}^{(j)}$  of  $\text{Max}(\mathcal{N}^{(j)})$  and an integer  $s$  such that

- $e_i \text{Max}(\mathcal{N}^{(j)})_r = u^{er} e_i \text{Max}(\mathcal{N}^{(j)})$ ,
- $\varphi_r(\sum_{i=0}^{d-1} u^{er} e_i \tilde{n}^{(j)}) = \lambda^{(j)} \tilde{n}^{(j)}$ ,
- $\hat{g}(\tilde{n}^{(j)}) = \sum_{i=0}^{d-1} (\omega_d(g)^{p^{i+j}s} \otimes 1) e_i \tilde{n}^{(j)}$  for all  $g \in \text{Gal}(K/K_0)$ , and
- $N(\tilde{n}^{(j)}) = 0$ .

(Here we use that  $T_{\text{st}}^{*,r}(\mathcal{N}^{(j)}) \cong \mathbb{F}(\chi^{p^j})$ .) The same lemma shows that

$$(3.3.9) \quad p^{i+j} s \equiv x_i^{(j)} - \frac{er}{p-1} + \frac{r_i^{(j)} p^d + r_{i+1}^{(j)} p^{d-1} + \dots + r_{i+d-1}^{(j)}}{e} \pmod{e}.$$

Recall from the proof of Lemma 3.3.3 that the natural map

$$\mathcal{N}^{(j)} \rightarrow \text{Max}(\mathcal{N}^{(j)})$$

sends  $n^{(j)}$  to  $\sum_i u^{t_i^{(j)}} e_i \tilde{n}^{(j)}$  for certain  $t_i^{(j)} \in [0, \frac{erp}{p-1}]$ . In particular, it follows that

$$(3.3.10) \quad \text{im}(\beta \circ \alpha) \supset \bigoplus_{j=0}^{d-1} u^{erp/(p-1)} \text{Max}(\mathcal{N}^{(j)}).$$

By Lemma 3.3.6 we may assume that  $m^{(j)} \in \varphi_r(\mathcal{M}_r)$ . (Note that we may arrange that  $\hat{g}(m^{(j)}) = (\omega_d(g)^{a_j} \otimes 1)m^{(j)}$ : for each  $i$  we have  $e_i(\mathcal{M}/u\mathcal{M}) \cong \bigoplus_j \mathbb{F}(\omega_d(g)^{a_j})$  as  $\mathbb{F}[\text{Gal}(K/K_0)]$ -module by our assumption on the descent data of  $\mathcal{M}$ , so we may permute the  $e_i m^{(j)}$  such that  $\hat{g}(e_i m^{(j)}) = \omega_d(g)^{a_j} e_i m^{(j)}$  for all  $j$ .) It follows that

$$\beta(m^{(j)}) \in \bigoplus_l \varphi_r((\text{Max } \mathcal{N}^{(l)})_r) = \bigoplus_l (k \otimes_{\mathbb{F}_p} \mathbb{F})[u^p]/u^{ep} \cdot \tilde{n}^{(l)}.$$

Since  $\beta$  commutes with descent data, it furthermore follows that

$$\beta(e_0 m^{(j)}) = \sum_{l=0}^{d-1} \gamma_{jl} u^{b_{jl}} e_0 \tilde{n}^{(l)},$$

where  $\gamma_{jl} \in \mathbb{F}$  and  $0 \leq b_{jl} < ep$  is uniquely determined by the congruences

$$(3.3.11) \quad \begin{aligned} b_{jl} &\equiv 0 \pmod{p}, \\ b_{jl} &\equiv a_j - p^l s \pmod{e}. \end{aligned}$$

It will be convenient to assume in the following that the  $a_j$ , and hence the  $x_i^{(j)}$ , are divisible by  $p$ . (Note that so far that only their values modulo  $e$  mattered and that  $(e, p) = 1$ .) Since  $r_i^{(j)} \in [0, er]$  it follows from (3.3.9) that

$$(3.3.12) \quad p^{i+j-1} s \text{ is congruent to an element of } \left[ \frac{x_i^{(j)}}{p} - \frac{er}{p-1}, \frac{x_i^{(j)}}{p} \right] \text{ modulo } e.$$

Our assumption on the  $a_j$  implies that two such intervals are either equal or disjoint. In particular,  $x_i^{(j)}$  only depends on  $i + j$  modulo  $d$ .

Suppose now that the  $(x_i^{(0)})_{i=0}^{d-1}$  are not all distinct. (This is the descent data on  $\mathcal{N}$ .) Then the  $(x_0^{(l)})_{l=0}^{d-1}$  are not all distinct. So there is  $j$  such that  $x_0^{(l)} \neq a_j$  for all  $l$ . From equations (3.3.11) and (3.3.12) we find that  $\frac{b_{jl}}{p}$  is congruent to an element of

$$\left[ \frac{a_j - x_0^{(l-1)}}{p}, \frac{a_j - x_0^{(l-1)}}{p} + \frac{er}{p-1} \right]$$

modulo  $e$ . Our assumption on the  $a_j$  then implies that  $b_{jl} > \frac{erp}{p-1}$  for all  $l$ . But, by applying Lemma 3.3.7 to the map  $\beta : e_0 \mathcal{M} \rightarrow \bigoplus_j e_0 \text{Max}(\mathcal{N}^{(j)})$  and using that  $r < p - 1$ , we see that this contradicts (3.3.10).  $\square$

**Theorem 3.3.13.** *Let  $\rho : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\overline{\mathbb{Q}_p})$  be a potentially semistable representation with Hodge–Tate weights  $-2, -1,$  and  $0$  for which  $\bar{\rho}$  is irreducible.*

*In the following we assume that  $a, b, c$  are integers satisfying  $a - b > 2,$   $b - c > 2,$   $a - c < p - 3.$*

- (i) *Suppose that  $\mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}} \cong \tilde{\omega}^a \oplus \tilde{\omega}^b \oplus \tilde{\omega}^c.$  Then  $\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \psi \oplus \psi^p \oplus \psi^{p^2},$  where*

$$\psi = \omega_3^{(a+a_0)+p(c+a_2)+p^2(b+a_1)}$$

or

$$\psi = \omega_3^{(a+2-a_2)+p(b+2-a_1)+p^2(c+2-a_0)}$$

with  $(a_0, a_1, a_2) \in \{(1, 1, 1), (1, 2, 0), (2, 1, 0)\}.$

- (ii) *Suppose that  $\mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}} \cong \tilde{\omega}_3^{a+pb+p^2c} \oplus \tilde{\omega}_3^{b+pc+p^2a} \oplus \tilde{\omega}_3^{c+pa+p^2b}.$  Then  $\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \psi \oplus \psi^p \oplus \psi^{p^2},$  where*

$$\psi = \omega_3^{(a+a_0)+p(c+a_2)+p^2(b+a_1)}$$

with  $(a_0, a_1, a_2) \in \{(0, 2, 1), (1, 1, 1), (1, 2, 0)\},$  or

$$\psi = \omega_3^{(a+a_0)+p(b+a_2)+p^2(c+a_1)}$$

with  $a_0, a_1, a_2 \in [0, 2]$  such that  $a_1 + a_2 + a_3 = 3.$

- (iii) *Suppose that  $\mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}} \cong \tilde{\omega}_3^{c+pb+p^2a} \oplus \tilde{\omega}_3^{b+pa+p^2c} \oplus \tilde{\omega}_3^{a+pc+p^2b}.$  Then  $\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \psi \oplus \psi^p \oplus \psi^{p^2},$  where*

$$\psi = \omega_3^{(c+2-a_0)+p(a+2-a_2)+p^2(b+2-a_1)}$$

with  $(a_0, a_1, a_2) \in \{(0, 2, 1), (1, 1, 1), (1, 2, 0)\},$  or

$$\psi = \omega_3^{(c+2-a_0)+p(b+2-a_2)+p^2(a+2-a_1)}$$

with  $a_0, a_1, a_2 \in [0, 2]$  such that  $a_1 + a_2 + a_3 = 3.$

Here we write  $\omega$  for  $\omega_1,$  the mod  $p$  cyclotomic character, and  $\tilde{\omega}$  for its Teichmüller lift. Note that we consider  $\tilde{\omega}_3$  as  $\overline{\mathbb{Q}_p}$ -valued character and  $\omega_3$  as  $\overline{\mathbb{F}_p}$ -valued character. This depends on a choice of embedding  $K_0 \rightarrow \overline{\mathbb{Q}_p},$  but all the statements in the theorem are independent of that choice.

*Proof.* As before we let  $K' = K_0$  be the unique unramified cubic extension of  $\mathbb{Q}_p$  and let  $K = K_0((-p)^{1/(p^3-1)}).$  We may assume (by a standard Baire category argument) that  $\rho : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\mathcal{O}_E)$  for some finite extension  $E/\mathbb{Q}_p.$  (Take  $E$  large enough so as to contain the images of all embeddings  $K \rightarrow \overline{\mathbb{Q}_p}.)$  The assumptions on  $a, b, c$  show that  $p > 7,$  so we may apply Proposition 3.3.1 with  $n = d = 3$  and  $r = 2 < p - 1$  to  $\rho|_{G_{K_0}},$  we see that there is an object  $\mathcal{M}$  of  $\mathbb{F}\text{-BrMod}_{\mathrm{dd}}^2$  with  $\mathrm{T}_{\mathrm{st}}^{*,2}(\mathcal{M}) \cong \bar{\rho},$  and that  $\mathrm{Gal}(K/K_0)$  acts on some basis of  $\mathcal{M}$  via the characters  $\omega^a \otimes 1, \omega^b \otimes 1, \omega^c \otimes 1$  in the first case, or  $\omega_3^{a+pb+p^2c} \otimes 1, \omega_3^{b+pc+p^2a} \otimes 1, \omega_3^{c+pa+p^2b} \otimes 1$  in the second case, or  $\omega_3^{c+pb+p^2a} \otimes 1, \omega_3^{b+pa+p^2c} \otimes 1, \omega_3^{a+pc+p^2b} \otimes 1$  in the third case. (Note that, for example in the first case, we at first get a basis on which  $\mathrm{Gal}(K/K_0)$  acts

via  $1 \otimes \sigma_0 \omega^a$ ,  $1 \otimes \sigma_0 \omega^b$ ,  $1 \otimes \sigma_0 \omega^c$ , but then we can just permute the induced basis on each  $e_i \mathcal{M}$  to get the desired basis for  $\mathcal{M}$ .)

Since  $\bar{\rho}$  is assumed to be irreducible,  $\bar{\rho}|_{G_{K_0}} \cong \chi \oplus \chi^p \oplus \chi^{p^2}$  for some character  $\chi : G_{K_0} \rightarrow \mathbb{F}^\times$  (e.g. by Lemma 2.16 of [ADP02]). By Corollary 3.2.9, there is a rank one submodule  $\mathcal{N}$  of  $\mathcal{M}$  with  $T_{\text{st}}^{*,2}(\mathcal{N}) \cong \chi$ . Then by Lemma 3.3.2  $\mathcal{N}$  has the form

- $\mathcal{N} = ((k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}) \cdot n$ ,
- $e_i \mathcal{N}_2 = u^{r_i} e_i \mathcal{N}$ ,
- $\varphi_2(\sum_{i=0}^2 u^{r_i} e_i n) = \lambda n$  for some  $\lambda \in (k \otimes_{\mathbb{F}_p} \mathbb{F})^\times$ , and
- $\hat{g}(n) = (\sum_{i=0}^2 (\omega_3(g)^{k_i} \otimes 1) e_i) n$  for all  $g \in \text{Gal}(K/K_0)$ ,

where each  $k_i \in \mathbb{Z}$ ,  $r_i \in [0, 2e]$ , and  $r_i \equiv p^2 k_{i+1} - k_i \pmod{e}$ . Note that  $e = p^3 - 1$ . We now consider the three cases in the statement of the theorem separately.

*First case:* if  $\text{WD}(\rho)|_{I_{\mathbb{Q}_p}} \cong \tilde{\omega}^a \oplus \tilde{\omega}^b \oplus \tilde{\omega}^c$ , then  $\text{WD}(\det \rho)|_{I_{K_0}} = \tilde{\omega}^{a+b+c}$ , so that  $(\text{WD}(\det \rho) \tilde{\omega}^{-(a+b+c)})|_{I_{K_0}} = 1$ . Thus  $(\det \rho)|_{G_{K_0}} \tilde{\omega}^{-(a+b+c)}$  is a crystalline character with all Hodge–Tate weights equal to  $-3$  (since  $\rho$  is assumed to have Hodge–Tate weights  $-2$ ,  $-1$  and  $0$ ), so we see that  $(\det \rho) \tilde{\omega}^{-(a+b+c)}|_{I_{K_0}} = \varepsilon^3$ , and in particular

$$\det \bar{\rho}|_{I_{K_0}} = \chi^{1+p+p^2}|_{I_{K_0}} = \omega^{a+b+c+3}.$$

By Proposition 3.3.8 and the assumptions on  $a, b, c$  we may assume that the  $k_i$  are a permutation of  $(1+p+p^2)a$ ,  $(1+p+p^2)b$ ,  $(1+p+p^2)c$ . (Consider  $\mathcal{N}/u\mathcal{N}$  to compare the  $k_i$  with the  $a_{\pi(i)}$  in the proposition.) Write  $k_i = (1+p+p^2)x_i$  with  $x_i \in \{a, b, c\}$ , so that  $r_i \equiv p^2 k_{i+1} - k_i \equiv (1+p+p^2)(x_{i+1} - x_i) \pmod{e}$ . Since  $r_i \in [0, 2e]$  and by our conditions on  $(a, b, c)$ , we can write  $r_i = (1+p+p^2)(x_{i+1} - x_i) + a_i e$  with  $a_i \in [0, 2]$ . By Lemma 3.3.2, we see that one of  $\chi|_{I_{K_0}}$ ,  $\chi^p|_{I_{K_0}}$  and  $\chi^{p^2}|_{I_{K_0}}$  is equal to

$$\omega_3^{(x_0+a_0)+p(x_2+a_2)+p^2(x_1+a_1)}.$$

Since  $\chi^{1+p+p^2}|_{I_{K_0}} = \omega^{a+b+c+3}$ , we see that

$$x_0 + x_1 + x_2 + a_0 + a_1 + a_2 \equiv a + b + c + 3 \pmod{p-1},$$

so  $a_0 + a_1 + a_2 = 3$ . By cyclic symmetry (our choice of  $\sigma_0$ ) we may assume that either  $x_0 = a$ ,  $x_1 = b$ ,  $x_2 = c$  or  $x_0 = a$ ,  $x_1 = c$ ,  $x_2 = b$ . Our conditions on  $(a, b, c)$  and that each  $r_i \in [0, 2e]$  then give the claimed result.

*Second case:* if  $\text{WD}(\rho)|_{I_{\mathbb{Q}_p}} \cong \tilde{\omega}_3^{a+pb+p^2c} \oplus \tilde{\omega}_3^{b+pc+p^2a} \oplus \tilde{\omega}_3^{c+pa+p^2b}$ , then arguing as in the previous case we see that  $\chi^{1+p+p^2}|_{I_{K_0}} = \omega^{a+b+c+3}$ , and by Proposition 3.3.8 and the assumptions on  $a, b, c$  we may assume that the  $k_i$  are a permutation of  $a + pb + p^2c$ ,  $b + pc + p^2a$ ,  $c + pa + p^2b$ . (To check the condition on the  $a_j$  in the proposition, it suffices to show that  $2(p^2 + p + 1) < p^2x + py + z < (p-3)(p^2 + p + 1)$  whenever  $x, y, z$  are in  $\{\pm(a-b), \pm(b-c), \pm(c-a)\}$  and  $x > 0$ . The second inequality is obvious. For the first, note that  $x \geq 3$  and that  $y$  and  $z$  are bounded below by  $-(p-4)$ .)

Write  $k_i = x_i + px'_i + p^2x''_i$ , where  $(x_i, x'_i, x''_i)$  is a cyclic permutation of  $(a, b, c)$ . So by our conditions on  $(a, b, c)$  and since  $r_i \in [0, 2e]$ , we can write  $r_i = (x'_{i+1} + px''_{i+1} + p^2x_{i+1}) - (x_i + px'_i + p^2x''_i) + a_i e$  with  $a_i \in [0, 2]$ . By Lemma 3.3.2, we see that one of  $\chi|_{I_{K_0}}$ ,  $\chi^p|_{I_{K_0}}$  and  $\chi^{p^2}|_{I_{K_0}}$  is equal to

$$\omega_3^{(x_0+a_0)+p(x_2+a_2)+p^2(x_1+a_1)}.$$

Just as above we see that  $a_0 + a_1 + a_2 = 3$ . Again we may assume that either  $x_0 = a, x_1 = b, x_2 = c$  or  $x_0 = a, x_1 = c, x_2 = b$ . The possibilities for the  $a_i$  are determined by the signs of  $x_{i+1} - x''_i$ . So in the first case we find that  $a_0 \neq 2, a_1 \neq 0$ , and  $a_2 \neq 2$ , whereas in the second case we get no restriction on the  $a_i$ .

*Third case:* this is analogous to the second case, or can be deduced from the second case by duality.  $\square$

#### 4. ABSTRACT FRAMEWORK

In order to avoid too much notational complexity, and with an eye to future applications and possible generalisations of our work, in this section we set up an axiomatic framework for the general setting in which our results will apply. In Section 7 we will establish that our axioms hold in the particular case of spaces of automorphic forms on definite unitary groups.

It will be convenient to introduce our axiomatic framework in two stages. Since our weight cycling arguments take place in the context of representations over  $\overline{\mathbb{F}}_p$  of the group  $\mathrm{GL}_n(\mathbb{Q}_p)$ , it is most natural to place our framework in the setting of such  $\overline{\mathbb{F}}_p$ -representations. This is what we do in Subsection 4.2 below. On the other hand, in practice, our axioms will be established by a comparison between a mod  $p$  and a characteristic zero setting, and it will be convenient to axiomatise this argument as well. The relevant axiomatic framework for this argument is the subject of Subsection 4.3, and the argument itself is the subject of Subsections 4.4 and 4.5. In Subsection 4.1 we recall some basic facts and notation related to representations of  $\mathrm{GL}_n$ .

**4.1. Some representations of  $\mathrm{GL}_n$ .** Let  $F$  be a number field. From now on until Section 6 we fix a place  $w$  of  $F$  lying over  $p$ . Fix a uniformiser  $\varpi$  of  $F_w$  and write  $k_w$  for the residue field of  $F_w$ .

**4.1.1. Dual Weyl modules.** Let  $\mathbb{Z}_+^n$  denote the set of tuples  $(\lambda_1, \dots, \lambda_n)$  of integers with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Let  $B_n \subset \mathrm{GL}_n$  be the Borel subgroup of upper-triangular matrices and let  $T_n \subset B_n$  be the diagonal maximal torus. We can then view any  $\lambda \in \mathbb{Z}_+^n$  as a dominant element of  $X^*(T_n) = \mathrm{Hom}(T_n, \mathbb{G}_m)$ , and let  $M_\lambda$  be the algebraic  $\mathbb{Z}$ -representation of  $\mathrm{GL}_n$  given by

$$M_\lambda := \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/\mathbb{Z}},$$

where  $w_0$  is the longest element of the Weyl group (see [Jan03] for more details of these notions). Then for any commutative ring  $A$  we have that

$M_\lambda(A) \cong M_\lambda(\mathbb{Z}) \otimes_{\mathbb{Z}} A$  is a finite free  $A$ -module with a natural action of  $\mathrm{GL}_n(A)$  that is functorial in  $A$ . For any  $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)}$ , let  $W_\lambda$  be the finite free  $\overline{\mathbb{Z}}_p$ -module with an action of  $\mathrm{GL}_n(\mathcal{O}_{F_w})$  given by

$$W_\lambda := \bigotimes_{\tau: F_w \rightarrow \overline{\mathbb{Q}}_p} (M_{\lambda_\tau}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}, \tau} \overline{\mathbb{Z}}_p).$$

Note that  $W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$  has a natural  $\mathrm{GL}_n(F_w)$ -action, as  $M_{\lambda_\tau}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}} F_w \cong M_{\lambda_\tau}(F_w)$ .

4.1.2. *Weights of  $\mathrm{GL}_n(k_w)$ .* We now explicitly describe all weights (i.e. irreducible  $\overline{\mathbb{F}}_p$ -representations) of  $\mathrm{GL}_n(k_w)$ .

Given any  $a \in \mathbb{Z}_+^n$  we define the algebraic  $\overline{\mathbb{F}}_p$ -representation  $N_a$  of  $\mathrm{GL}_n$  to be the subrepresentation of  $M_{a/\overline{\mathbb{F}}_p}$  that is generated by the highest weight vector. (It is in fact the unique irreducible subrepresentation of  $M_{a/\overline{\mathbb{F}}_p}$ ; see §II.2 in [Jan03].)

We say that an element  $a \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)}$  is a *restricted weight* if for each  $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  and each  $1 \leq i \leq n-1$  we have  $a_{\sigma, i} - a_{\sigma, i+1} \leq p-1$ . For a restricted weight  $a \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)}$ , we define an  $\overline{\mathbb{F}}_p$ -representation  $F_a$  of  $\mathrm{GL}_n(k_w)$  by

$$(4.1.3) \quad F_a := \bigotimes_{\sigma: k_w \rightarrow \overline{\mathbb{F}}_p} (N_{a_\sigma}(k_w) \otimes_{k_w, \sigma} \overline{\mathbb{F}}_p).$$

It is irreducible, and every irreducible  $\overline{\mathbb{F}}_p$ -representation of  $\mathrm{GL}_n(k_w)$  is of the form  $F_a$  for some  $a$ . For restricted weights  $a$  and  $b$  we have  $F_a \cong F_b$  as representations of  $\mathrm{GL}_n(k_w)$  if and only if for each  $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$ , we have

$$a_{\sigma, i} - a_{\sigma, i+1} = b_{\sigma, i} - b_{\sigma, i+1}$$

for each  $1 \leq i \leq n-1$  and the character  $k_w^\times \rightarrow \overline{\mathbb{F}}_p^\times$  given by

$$x \mapsto \prod_{\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)} \sigma(x)^{a_{\sigma, n} - b_{\sigma, n}}$$

is trivial. (For all this, see for example Theorems 3.9 and 3.10 of and the appendix to [Her09].)

If  $F_w/\mathbb{Q}_p$  is unramified, the sets  $\mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)$  and  $\mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  are in natural bijection. We can thus define  $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)}$  by demanding that  $\lambda_\tau = a_\tau$  for all  $\tau$  and we call it the *lift* of  $a$ . Since  $N_a$  is by definition a subrepresentation of  $M_{a/\overline{\mathbb{F}}_p}$ , it follows that we have a natural  $\mathrm{GL}_n(k_w)$ -linear map  $N_a(k_w) \hookrightarrow M_a(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}} k_w$  and hence a natural  $\mathrm{GL}_n(k_w)$ -linear map  $F_a \hookrightarrow W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$ .

4.1.4. *Locally algebraic modules.* If  $G$  is an open subgroup of  $\mathrm{GL}_n(F_w)$ , we say that a function  $f : G \rightarrow \overline{\mathbb{Q}}_p$  is *locally algebraic* if each element of  $G$  has a neighbourhood  $U$  such that the restriction  $f|_U$  of  $f$  to  $U$  is of the form  $f|_U : g \mapsto \phi(g)$ , for some element  $\phi$  of the affine coordinate ring of the affine algebraic group  $(\mathrm{Res}_{F_w/\mathbb{Q}_p} \mathrm{GL}_n)/\overline{\mathbb{Q}}_p$ .

If  $V$  is a finite free  $\overline{\mathbb{Z}}_p$ -module equipped with an action of an open subgroup  $G$  of  $\mathrm{GL}_n(F_w)$ , we say that the  $G$ -action on  $V$  is *locally algebraic* if each of the matrix coefficients of  $V$  is a locally algebraic function on  $G$ . (Recall that the matrix coefficients of  $V$  are the functions on  $G$  of the form  $g \mapsto \langle gv, v^\vee \rangle$ , where  $v \in V$  and  $v^\vee \in V^\vee$ .)

As an example, note that the  $\mathrm{GL}_n(\mathcal{O}_{F_w})$ -action on each of the dual Weyl modules  $W_\lambda$  introduced in §4.1.1 is locally algebraic.

4.2. **Axioms for mod  $p$  representations.** We keep our notation of the preceding subsection. In particular, recall that we assume that  $w$  is a place of the number field  $F$  that lies over  $p$ . From now on until Section 6 we will assume moreover that  $F_w/\mathbb{Q}_p$  is unramified.

Let  $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$  be a continuous representation such that  $\bar{r}|_{G_{F_w}}$  is irreducible. Let  $\mathcal{P}$  be a set of finite places of  $F$  which do not lie over  $p$ , and let  $\mathbb{T}^{\mathcal{P}}$  denote the commutative  $\overline{\mathbb{Z}}_p$ -polynomial algebra generated by formal variables  $T_v^{(j)}$  where  $1 \leq j \leq n$  and  $v \in \mathcal{P}$ . We will write  $\mathbb{T}$  for  $\mathbb{T}^{\mathcal{P}}$  when the choice of  $\mathcal{P}$  is clear.

Assume that  $\bar{r}$  is unramified at each place  $v \in \mathcal{P}$ . We define a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}$  with residue field  $\overline{\mathbb{F}}_p$  by demanding that for each  $v \in \mathcal{P}$ , the characteristic polynomial of  $\bar{r}^\vee(\mathrm{Frob}_v)$  is equal to the reduction modulo  $\mathfrak{m}$  of

$$X^n + \cdots + (-1)^j (\mathbf{N}v)^{j(j-1)/2} T_v^{(j)} X^{n-j} + \cdots + (-1)^n (\mathbf{N}v)^{n(n-1)/2} T_v^{(n)}.$$

(By definition,  $\mathfrak{m}$  is determined by  $\bar{r}$ , but in the applications  $\mathcal{P}$  will be chosen so that conversely  $\bar{r}$  is determined by  $\mathfrak{m}$ .)

We let  $V$  denote a weight of the group  $\mathrm{GL}_n(k_w)$ . (Recall that when  $n = 3$  and  $k_w = \mathbb{F}_p$  such a weight is determined by a triple of integers  $x \geq y \geq z$  satisfying  $x - y, y - z \leq p - 1$  and we will write  $V = F(x, y, z)$ . For more details of this, see §4.1.2.)

Write  $G = \mathrm{GL}_n$ , and consider the Hecke algebra  $\mathcal{H}_G(V)$  defined in Section 2.2. In Section 2.2 we defined the elements  $\bar{T}_{\mu, \varpi} \in \mathcal{H}_G(V)$  for  $\mu \in X_*(T_n)_-$ , which we can think of as a non-decreasing sequence of  $n$  integers. For  $1 \leq j \leq n$  we let  $\bar{T}_j = \bar{T}_{\lambda_j, \varpi}$ , where  $\lambda_j = (0, \dots, 0, 1, \dots, 1)$ , with  $n - j$  zeroes followed by  $j$  ones.

Now let  $S$  be an  $\overline{\mathbb{F}}_p$ -vector space with an action of  $\mathbb{T}$ , and suppose that  $S$  additionally carries a smooth action of  $\mathrm{GL}_n(F_w)$ , and that the actions of  $\mathbb{T}$  and  $\mathrm{GL}_n(F_w)$  commute. If  $V$  is a weight of  $\mathrm{GL}_n(k_w)$ , we write  $S(V)$  for  $(V \otimes_{\overline{\mathbb{F}}_p} S)^{\mathrm{GL}_n(\mathcal{O}_{F_w})}$ , so that  $S(V)$  carries commuting actions of  $\mathbb{T}$  and  $\mathcal{H}_G(V)$ . We additionally make the following assumptions, letting  $F_{w,n}$  denote the unramified extension of  $F_w$  of degree  $n$  and  $k_{w,n}$  its residue field.

- A1 *Hecke operators at  $p$  act by zero.* For all weights  $V$  of  $\mathrm{GL}_n(k_w)$  the vector space  $S(V)$  is finite dimensional and each of the Hecke operators  $\bar{T}_1, \dots, \bar{T}_{n-1}$  acts nilpotently on  $S(V)_\mathfrak{m}$ .
- A2 *Lifts of specified weight and type.* Suppose that  $F_w = \mathbb{Q}_p$  and  $n = 3$ . Suppose that  $S(V)_\mathfrak{m} \neq 0$  for some  $V = F(x, y, z)$ . Then
- $\bar{r}|_{G_{F_w}}$  has a crystalline lift with Hodge–Tate weights  $-(x+2), -(y+1), -z$ .
  - Suppose that  $F(x, y, z)$  is a Jordan–Hölder factor of the reduction mod  $p$  of the cuspidal representation  $R_{\mathbb{T}}^\theta$  of  $\mathrm{GL}_3(k_w)$  as in Subsection 2.4, for some primitive character  $\theta : k_{w,3}^\times \rightarrow \overline{\mathbb{Q}_p}^\times$ . Then  $\bar{r}|_{G_{F_w}}$  has a potentially semistable lift with Hodge–Tate weights  $-2, -1, 0$  and inertial type

$$\bigoplus_{\sigma \in \mathrm{Gal}(k_{w,3}/k_w)} \sigma(\theta \circ \mathrm{Art}_{F_w,3}^{-1}).$$

- Suppose that  $F(x, y, z)$  is a Jordan–Hölder factor of the reduction mod  $p$  of the principal series representation

$$\mathrm{Ind}_{B_3(k_w)}^{\mathrm{GL}_3(k_w)} (\chi_1 \otimes \chi_2 \otimes \chi_3)$$

of  $\mathrm{GL}_3(k_w)$  as in Subsection 2.4, for some distinct characters  $\chi_i : k_w^\times \rightarrow \overline{\mathbb{Q}_p}^\times$ . Then  $\bar{r}|_{G_{F_w}}$  has a potentially semistable lift with Hodge–Tate weights  $-2, -1, 0$  and inertial type

$$\bigoplus_{i=1}^3 \chi_i \circ \mathrm{Art}_{F_w}^{-1}.$$

*Remark 4.2.1.* We could have stated Axiom A2 in greater generality, but we decided to give the simplified statement in the case that  $n = 3$  and  $F_w = \mathbb{Q}_p$  because this is all we need in our main theorems.

*Definition 4.2.2.* We will say that  $\bar{r}$  is *modular of weight  $V$*  if  $S(V)_\mathfrak{m} \neq 0$ . We let  $W_w(\bar{r})$  denote the set of weights  $V$  for which  $\bar{r}$  is modular of weight  $V$ .

In the eventual application to automorphic forms on definite unitary groups, Axioms A1 and A2 will be consequences of local-global compatibility at places dividing  $p$  for the  $p$ -adic Galois representations associated to automorphic forms.

**4.3. Axioms for characteristic zero representations.** We maintain the notation of the preceding subsection. In this subsection we introduce an axiomatic framework that relates certain characteristic  $p$  and characteristic 0 representations of  $\mathrm{GL}_n(F_w)$ .

To this end, we suppose given a  $\overline{\mathbb{Z}_p}$ -module  $\tilde{S}$  and an  $\overline{\mathbb{F}_p}$ -vector space  $S$ , each equipped with commuting actions of  $\mathrm{GL}_n(F_w)$  and  $\mathbb{T}$ , together with an

embedding

$$(4.3.1) \quad S \hookrightarrow \tilde{S} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p.$$

that is equivariant for the two actions. Suppose that the  $\mathrm{GL}_n(F_w)$ -action on  $S$  is smooth, so  $S$  is as in the preceding section. If  $\tilde{V}$  is a  $\overline{\mathbb{Z}}_p$ -module equipped with a  $\mathrm{GL}_n(\mathcal{O}_{F_w})$ -action, then we write  $\tilde{S}(\tilde{V}) = (\tilde{V} \otimes_{\overline{\mathbb{Z}}_p} \tilde{S})^{\mathrm{GL}_n(\mathcal{O}_{F_w})}$ . Hence if  $\tilde{V}$  is an  $A$ -module for a  $\overline{\mathbb{Z}}_p$ -algebra  $A$  equipped with a  $\mathrm{GL}_n(\mathcal{O}_{F_w})$ -action, then  $\tilde{S}(\tilde{V})$  is an  $A$ -module as well.

For  $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)}$ , we defined in §4.1.1 a dual Weyl module  $W_\lambda$ . It is a finite free  $\overline{\mathbb{Z}}_p$ -module with a (locally) algebraic  $\mathrm{GL}_n(\mathcal{O}_{F_w})$ -action. Recall that  $W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$  has a natural  $\mathrm{GL}_n(F_w)$ -action. Thus the  $\overline{\mathbb{Q}}_p$ -vector space

$$\tilde{S}(W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p) \cong ((W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Z}}_p} \tilde{S})^{\mathrm{GL}_n(\mathcal{O}_{F_w})}$$

carries a natural action of the (commutative) double coset algebra

$$\overline{\mathbb{Z}}_p[\mathrm{GL}_n(\mathcal{O}_{F_w}) \backslash \mathrm{GL}_n(F_w) / \mathrm{GL}_n(\mathcal{O}_{F_w})]$$

that commutes with the  $\mathbb{T}$ -action.

**Definition 4.3.2.** For  $\mu \in X_*(T_n)_-$  denote by  $T_{\mu, w}$  the following double coset operator on  $\tilde{S}(W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p)$ :

$$T_{\mu, w} := [\mathrm{GL}_n(\mathcal{O}_{F_w})\mu(\varpi)\mathrm{GL}_n(\mathcal{O}_{F_w})].$$

Moreover, let  $T_j = T_{\lambda_j, w}$ , where  $\lambda_j = (0, \dots, 0, 1, \dots, 1)$ , with  $n - j$  zeroes followed by  $j$  ones.

We introduce the following axioms related to the above set-up.

**Å1 Finiteness and extension of scalars.** If  $\tilde{V}$  is a free finite-rank  $\overline{\mathbb{Z}}_p$ -module equipped with a locally algebraic action of  $\mathrm{GL}_n(\mathcal{O}_{F_w})$ , then  $\tilde{S}(\tilde{V})$  is a finite free  $\overline{\mathbb{Z}}_p$ -module and for  $A = \overline{\mathbb{Q}}_p$  and  $A = \overline{\mathbb{F}}_p$  the natural map

$$\tilde{S}(\tilde{V}) \otimes_{\overline{\mathbb{Z}}_p} A \rightarrow \tilde{S}(\tilde{V} \otimes_{\overline{\mathbb{Z}}_p} A)$$

is an isomorphism.

**Å2 Crystalline lifts.** Suppose that  $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F_w, \overline{\mathbb{Q}}_p)}$ . If  $\tilde{S}(W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p)_\mathfrak{m} \neq 0$ , then  $\bar{r}|_{G_{F_w}}$  has a crystalline lift  $\rho : G_{F_w} \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$  such that for all  $\tau : F_w \rightarrow \overline{\mathbb{Q}}_p$  we have  $\mathrm{HT}_\tau(\rho) = -(\lambda_{\tau, 1} + n - 1, \dots, \lambda_{\tau, n})$ .

If moreover  $T_j$  has eigenvalue  $t_j \in \overline{\mathbb{Q}}_p$  on  $\tilde{S}(W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p)_\mathfrak{m}$  for  $1 \leq j \leq n$ , then we can demand that  $\varphi^{[F_w: \overline{\mathbb{Q}}_p]}$  acting on the  $\overline{\mathbb{Q}}_p$ -vector space  $D_{\mathrm{cris}}(\rho)$  has characteristic polynomial

$$(X^n + \dots + (-1)^j q_w^{j(j-1)/2} t_j X^{n-j} + \dots + (-1)^n q_w^{n(n-1)/2} t_n)^{[F_w: \overline{\mathbb{Q}}_p]},$$

where  $q_w = \#k_w = p^{[F_w: \overline{\mathbb{Q}}_p]}$ .

**Å3 Lifts of tame type.** Suppose that  $R$  is a cuspidal representation  $(-1)^{n-1}R_{\mathbb{T}}^{\theta}$  of  $\mathrm{GL}_n(k_w)$  for some primitive character  $\theta : k_{w,n}^{\times} \rightarrow \overline{\mathbb{Q}_p}^{\times}$ , respectively a principal series representation  $\mathrm{Ind}_{B_n(k_w)}^{\mathrm{GL}_n(k_w)}(\chi_1 \otimes \cdots \otimes \chi_n)$  of  $\mathrm{GL}_n(k_w)$  for some distinct characters  $\chi_i : k_w^{\times} \rightarrow \overline{\mathbb{Q}_p}^{\times}$ . (See Subsection 2.4 for definitions.)

If  $\tilde{S}(R)_{\mathfrak{m}} \neq 0$ , then  $\bar{r}|_{G_{F_w}}$  has a potentially semistable lift  $\rho : G_{F_w} \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$  such that for all  $\tau : F_w \rightarrow \overline{\mathbb{Q}_p}$  we have  $\mathrm{HT}_{\tau}(\rho) = -(n-1, \dots, 1, 0)$  and whose inertial type is

$$\bigoplus_{\sigma \in \mathrm{Gal}(k_{w,n}/k_w)} \sigma(\theta \circ \mathrm{Art}_{F_{w,n}}^{-1}),$$

respectively

$$\bigoplus_{i=1}^n \chi_i \circ \mathrm{Art}_{F_w}^{-1}.$$

**Theorem 4.3.3.** *If Axioms Å1–Å3 are satisfied by  $S$  and  $\tilde{S}$ , then  $S$  also satisfies Axioms A1–A2 of the preceding subsection.*

*Proof.* In Corollary 4.5.3 we verify that Å1 and Å2 together imply A1. If  $n = 3$ ,  $F_w = \mathbb{Q}_p$ , and  $a = (x, y, z)$  is restricted, let  $\lambda \in \mathbb{Z}_+^3$  denote the lift of  $a$ . From the embedding (4.4.1) we see that  $(\tilde{S}(W_{\lambda}) \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p})_{\mathfrak{m}} \neq 0$ , hence  $(\tilde{S}(W_{\lambda}) \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{Q}_p})_{\mathfrak{m}} \neq 0$ . From Å1 and Å2 we deduce that the first part of A2 holds. For the second and third parts of A2 we argue similarly, letting a  $\mathrm{GL}_n(k_w)$ -stable lattice  $R_0$  in the cuspidal (resp. principal series) representation  $R$  such that  $R_0 \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p}$  contains  $F_a$  as a subrepresentation play the role of  $W_{\lambda}$  and using axioms Å1 and Å3.  $\square$

**4.4. Hecke action at  $p$ .** We suppose that  $S$  and  $\tilde{S}$  are as in the preceding subsection. In this subsection we will only require Axiom Å1.

Let  $a \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(k_w, \overline{\mathbb{F}_p})}$  be a restricted weight and let  $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F_w, \overline{\mathbb{Q}_p})}$  denote its lift as in §4.1.2. From the natural injections  $F_a \hookrightarrow W_{\lambda} \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p}$  and  $S \hookrightarrow \tilde{S} \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p}$  we get a  $\mathbb{T}$ -equivariant embedding  $S(F_a) \hookrightarrow \tilde{S}(W_{\lambda} \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p})$ . By Axiom Å1 we can think of it as  $\mathbb{T}$ -equivariant embedding

$$(4.4.1) \quad S(F_a) \hookrightarrow \tilde{S}(W_{\lambda}) \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p}.$$

The goal of this subsection is to compare the action of the Hecke operators  $\overline{T}_{\mu, \varpi}$  on  $S(F_a)$  with the action of suitably normalised Hecke operators on  $\tilde{S}(W_{\lambda}) \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{Q}_p}$ . Note that this last space is isomorphic to  $\tilde{S}(W_{\lambda} \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{Q}_p})$  by Axiom Å1, so that the Hecke operators  $T_{\mu, w}$  of Definition 4.3.2 act on it.

**Proposition 4.4.2.** *Suppose that  $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F_w, \overline{\mathbb{Q}_p})}$  is the lift of the restricted weight  $a$ , and suppose that Axiom Å1 holds. The action of the Hecke*

operator  $(\prod_{\tau:F_w \rightarrow \overline{\mathbb{Q}}_p} \tau(\varpi)^{-\langle \mu, \lambda_\tau \rangle}) T_{\mu, w}$  on  $\tilde{S}(W_\lambda) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$  has the following properties:

- (i) It stabilises the  $\overline{\mathbb{Z}}_p$ -lattice  $\tilde{S}(W_\lambda)$ .
- (ii) The induced action on  $\tilde{S}(W_\lambda) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$  stabilises the subspace  $S(F_a)$ .
- (iii) The induced action on  $S(F_a)$  coincides with the action of the operator  $\overline{T}_{\mu, \varpi}$  in  $\mathcal{H}(F_a)$ .

*Proof.* Write  $\mathrm{GL}_n(\mathcal{O}_{F_w})\mu(\varpi)\mathrm{GL}_n(\mathcal{O}_{F_w}) = \coprod_i k_i \mu(\varpi) \mathrm{GL}_n(\mathcal{O}_{F_w})$  as a finite disjoint union with  $k_i \in \mathrm{GL}_n(\mathcal{O}_{F_w})$ . For

$$f \in \tilde{S}(W_\lambda \otimes \overline{\mathbb{Q}}_p) \cong ((W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Z}}_p} \tilde{S})^{\mathrm{GL}_n(\mathcal{O}_{F_w})}$$

we have

$$(4.4.3) \quad T_{\mu, w}(f)x = \sum_i k_i \mu(\varpi) f.$$

To establish part (i) it suffices to show that  $(\prod_{\tau:F_w \rightarrow \overline{\mathbb{Q}}_p} \tau(\varpi)^{-\langle \mu, \lambda_\tau \rangle}) \mu(\varpi)$  stabilises  $W_\lambda \subset W_\lambda \otimes \overline{\mathbb{Q}}_p$ . Recall that  $W_\lambda \cong \prod_{\tau:F_w \rightarrow \overline{\mathbb{Q}}_p} (M_{\lambda_\tau}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}, \tau} \overline{\mathbb{Z}}_p)$ . For  $\nu_\tau \in X^*(T_n)$  the element  $\mu(\varpi) \in T_n(F_w)$  acts on the  $\nu_\tau$ -weight space of  $M_{\lambda_\tau}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}, \tau} \overline{\mathbb{Q}}_p$  as the scalar  $\tau(\varpi)^{\langle \mu, \nu_\tau \rangle}$ . Since  $\nu_\tau \leq \lambda_\tau$  and  $\mu \in X_*(T_n)_-$ , we see that  $\langle \mu, \nu_\tau \rangle \geq \langle \mu, \lambda_\tau \rangle$ , so  $\tau(\varpi)^{-\langle \mu, \lambda_\tau \rangle} \mu(\varpi)$  stabilises  $M_{\lambda_\tau}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}, \tau} \overline{\mathbb{Z}}_p$  and part (i) follows.

To establish part (ii), since  $S \hookrightarrow \tilde{S} \otimes \overline{\mathbb{F}}_p$  is  $\mathrm{GL}_n(F_w)$ -equivariant, it suffices to show that  $(\prod_{\tau:F_w \rightarrow \overline{\mathbb{Q}}_p} \tau(\varpi)^{-\langle \mu, \lambda_\tau \rangle}) \mu(\varpi)$  stabilises  $F_a \subset W_\lambda \otimes \overline{\mathbb{F}}_p$ . Note that the induced action of  $\tau(\varpi)^{-\langle \mu, \lambda_\tau \rangle} \mu(\varpi)$  on the reduction  $M_{\lambda_\tau}(\mathcal{O}_{F_w}) \otimes_{\mathcal{O}_{F_w}, \tau} \overline{\mathbb{F}}_p$  is the linear projection onto the  $\nu_\tau$ -weight spaces with  $\langle \mu, \nu_\tau \rangle = \langle \mu, \lambda_\tau \rangle$ . Let  $\sigma : k_w \rightarrow \overline{\mathbb{F}}_p$  denote the embedding induced by  $\tau$ . Since  $N_{a_\sigma}$  has a weight space decomposition, as it is an algebraic subrepresentation of  $M_{\lambda_\tau/\mathbb{F}_p}$ , it follows that  $\tau(\varpi)^{-\langle \mu, \lambda_\tau \rangle} \mu(\varpi)$  stabilises the subspace  $N_{a_\sigma}(k_w) \otimes_{k_w, \sigma} \overline{\mathbb{F}}_p$ . This implies that part (ii) holds.

By comparing formulae (2.2.1) and (4.4.3) we see that part (iii) holds provided  $\overline{T}_{\mu, \varpi}(\mu(\varpi)) \in \mathrm{End}_{\overline{\mathbb{F}}_p}(F_a)$  is the linear projection onto the weight spaces  $\nu_\sigma$  of  $N_{a_\sigma}(k_w) \otimes_{k_w, \sigma} \overline{\mathbb{F}}_p$  such that  $\langle \mu, \nu_\sigma \rangle = \langle \mu, a_\sigma \rangle$  for all  $\sigma$ .

Fix an embedding  $\sigma_0 : k_w \rightarrow \overline{\mathbb{F}}_p$  and let  $\sigma_i := \sigma_0^{p^i}$  for any integer  $i$ . Let  $f := [k_w : \mathbb{F}_p]$ . By Steinberg's tensor product theorem (Corollary II.3.17 in [Jan03]) we have  $\bigotimes_{i=0}^{f-1} N_{a_{\sigma_i}}^{(i)} \cong N_A$ , where  $A := \sum_{i=0}^{f-1} p^i a_{\sigma_i}$  and where the superscript  $(i)$  denotes the  $i$ -th Frobenius twist (i.e. the composition with the  $i$ -th power of the Frobenius endomorphism of  $\mathrm{GL}_n/\mathbb{F}_p$ ). Therefore

$$F_a \cong \bigotimes_{\sigma} (N_{a_\sigma}(k_w) \otimes_{k_w, \sigma} \overline{\mathbb{F}}_p) \cong N_A(k_w) \otimes_{k_w, \sigma_0} \overline{\mathbb{F}}_p.$$

Recall from §2.2 that  $\overline{T}_{\mu, \varpi}(\mu(\varpi))$  projects onto the subspace of  $N_A$  that is invariant by the  $k_w$ -points of the unipotent radical of the parabolic of  $\mathrm{GL}_n$

defined by  $-\mu \in X_*(T_n)_+$ . It consists of the  $\nu$ -weight spaces of  $N_A$  satisfying  $\langle \mu, \nu \rangle = \langle \mu, A \rangle$  by Lemma 2.3 in [Her11a]. We can write  $\nu = \sum_{i=0}^{f-1} p^i \nu_{\sigma_i}$  where  $\nu_{\sigma_i}$  is a weight of  $N_{a_{\sigma_i}}$  and we see that  $\langle \mu, \nu \rangle = \langle \mu, A \rangle$  iff  $\langle \mu, \nu_{\sigma_i} \rangle = \langle \mu, a_{\sigma_i} \rangle$  for all  $i$ , which is what we wanted to prove.  $\square$

**Corollary 4.4.4.** *The action of  $(\prod_{\tau} \tau(\varpi)^{-\sum_{i=1}^j \lambda_{\tau, n+1-i}}) T_j$  on  $\tilde{S}(W_\lambda)$  induces the action of  $\bar{T}_j \in \mathcal{H}(F_a)$  on  $S(F_a) \subset \tilde{S}(W_\lambda) \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p$ .*

**4.5. A criterion for the vanishing of Hecke operators at  $p$ .** In this subsection we explain how one can deduce information about Hecke operators from information about the associated Galois representation via Axiom  $\tilde{A}2$ . In particular, we deduce that certain Hecke operators in characteristic  $p$  have all eigenvalues equal to 0 in the situations that we consider in this paper.

First we recall a variant of a lemma of Deligne and Serre ([DS74]).

**Lemma 4.5.1.** *Let  $A$  be a commutative  $\bar{\mathbb{Z}}_p$ -algebra acting on a finite-rank free  $\bar{\mathbb{Z}}_p$ -module  $M$ . If  $\mathfrak{m}$  is a maximal ideal of  $A$  such that  $(M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p)_{\mathfrak{m}} \neq 0$ , then there exists a homomorphism of  $\bar{\mathbb{Z}}_p$ -algebras  $\theta : A \rightarrow \bar{\mathbb{Q}}_p$  whose kernel is contained in  $\mathfrak{m}$ , such that the  $\theta$ -eigenspace of  $M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p$  is nonzero.*

*Proof.* Since  $(M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p)_{\mathfrak{m}} \neq 0$ , we see that  $M_{\mathfrak{m}} \neq 0$ , and hence that  $(M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p)_{\mathfrak{m}} \neq 0$ . Now  $M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p$  is a finite-dimensional  $\bar{\mathbb{Q}}_p$ -vector space, of which the localisation  $(M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p)_{\mathfrak{m}}$  is a subspace. (More precisely, it is the maximal subspace on which all of the elements  $a \notin \mathfrak{m}$  act invertibly.) In particular,  $(M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p)_{\mathfrak{m}}$  is again finite-dimensional.

If we let  $A'$  denote the image of  $A \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p$  in  $\text{End}_{\bar{\mathbb{Q}}_p}((M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p)_{\mathfrak{m}})$ , then  $A'$  is a finite-dimensional commutative  $\bar{\mathbb{Q}}_p$ -algebra, and hence admits a surjection  $\theta' : A' \rightarrow \bar{\mathbb{Q}}_p$ . Since  $A'$  acts faithfully on  $(M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p)_{\mathfrak{m}}$ , the  $\theta'$ -eigenspace of  $(M \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{Q}}_p)_{\mathfrak{m}}$  is then nonzero. If we let  $\theta : A \rightarrow \bar{\mathbb{Q}}_p$  denote the homomorphism obtained by composing  $\theta'$  with the natural map  $A \rightarrow A'$ , then  $\theta$  satisfies the conditions of the lemma.  $\square$

In the remainder of this subsection we will follow the analysis of Section 2.7 of [Ger09], which considers the ordinary situation. Let  $E$  be an algebraic extension of  $\mathbb{Q}_p$ . Let  $w|p$  be a place of  $F$ , and let  $\rho : G_{F_w} \rightarrow \text{GL}_n(E)$  be a crystalline representation. Assume that  $E$  contains the images of all embeddings  $F_w \hookrightarrow \bar{E}$ . Let  $D = D_{\text{cris}}(\rho) = (B_{\text{cris}} \otimes_{\mathbb{Q}_p} \rho)^{G_{F_w}}$ . This is a weakly admissible filtered  $\varphi$ -module; if  $F_w^0$  is the maximal absolutely unramified subfield of  $F_w$ , and  $\phi^0$  is the absolute Frobenius automorphism of  $F_w^0$ , then  $D$  is a finite free  $F_w^0 \otimes_{\mathbb{Q}_p} E$ -module of rank  $n$  with

- a  $\phi^0$ -semilinear,  $E$ -linear automorphism  $\varphi$ , and
- a separated and exhaustive decreasing filtration  $(\text{Fil}^i D_{F_w})_{i \in \mathbb{Z}}$  on the tensor product  $D_{F_w} := D \otimes_{F_w^0} F_w$  by  $F_w \otimes_{\mathbb{Q}_p} E$ -submodules.

For each embedding  $\sigma : F_w^0 \hookrightarrow E$ , we let  $D_\sigma := D \otimes_{F_w^0 \otimes_{E, \sigma} 1} E$ , so that  $D = \prod_{\sigma : F_w^0 \hookrightarrow E} D_\sigma$ . Similarly, for each embedding  $\tau : F_w \hookrightarrow E$ , we let  $D_{F_w, \tau} := D_{F_w} \otimes_{F_w \otimes_{E, \tau} 1} E$ , so that  $D_{F_w} = \prod_{\tau : F_w \hookrightarrow E} D_{F_w, \tau}$ . The filtration on  $D_{F_w}$  is induced by filtrations on each  $D_{F_w, \tau}$ , so that we may write  $\text{Fil}^i(D_{F_w}) = \prod_{\tau : F_w \hookrightarrow E} \text{Fil}^i(D_{F_w, \tau})$ . Then by definition  $\text{HT}_\tau(\rho)$  is  $\dim_E \text{gr}^i(D_{F_w, \tau})$ .

The map  $\varphi^{[F_w^0 : \mathbb{Q}_p]}$  induces an isomorphism of  $E$ -vector spaces  $\varphi^{[F_w^0 : \mathbb{Q}_p]} : D_\sigma \xrightarrow{\sim} D_\sigma$  for each  $\sigma$ . Assume now that  $E$  is a finite extension of  $\mathbb{Q}_p$ . Let  $D'$  be a  $\varphi$ -stable free  $F_w^0 \otimes_{\mathbb{Q}_p} E$ -submodule of  $D$ . We define the Hodge and Newton numbers of  $D'$  in the usual way, by forgetting the  $E$ -structure, and thinking of  $D'$  as a finite-dimensional  $F_w^0$ -vector space. Thus

$$t_H(D') = \sum_{i \in \mathbb{Z}} (\dim_{F_w} \text{gr}^i D'_{F_w}) i,$$

$$t_N(D') = \sum_{\alpha \in \mathbb{Q}} (\dim_{F_w^0} D'_\alpha) \alpha,$$

where  $D'_\alpha$  is the slope- $\alpha$  part of  $D'$ . Since  $D$  is weakly admissible, we have  $t_N(D') \geq t_H(D')$ , with equality if and only if there is a crystalline sub- $E$ -representation  $\rho'$  of  $\rho$  with  $D' = D_{\text{cris}}(\rho')$ .

In the following, we let  $\text{val}_p$  denote the valuation of  $\overline{\mathbb{Q}_p}$ , normalised such that  $\text{val}_p(p) = 1$ . We note that the next Proposition does *not* require the running assumption that the extension  $F_w/\mathbb{Q}_p$  is unramified, so we phrase the statement and proof without this assumption.

**Proposition 4.5.2.** *Let  $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F_w, \overline{\mathbb{Q}_p})}$ , and assume that  $\bar{r}|_{G_{F_w}}$  has a crystalline lift  $\rho : G_{F_w} \rightarrow \text{GL}_n(\overline{\mathbb{Q}_p})$  such that for all  $\tau : F_w \rightarrow \overline{\mathbb{Q}_p}$  we have  $\text{HT}_\tau(\rho) = -(\lambda_{\tau, 1} + n - 1, \dots, \lambda_{\tau, n})$ , and that  $\varphi^{[F_w : \mathbb{Q}_p]}$  acting on the  $\overline{\mathbb{Q}_p}$ -vector space  $D_{\text{cris}}(\rho)$  has characteristic polynomial*

$$(X^n + \dots + (-1)^j q_w^{j(j-1)/2} t_j X^{n-j} + \dots + (-1)^n q_w^{n(n-1)/2} t_n)^{[F_w : \mathbb{Q}_p]},$$

where  $q_w = \#k_w$ .

If for some  $1 \leq j \leq n - 1$  we have

$$\text{val}_p(t_j) = \frac{1}{[F_w : F_w^0]} \sum_{\tau : F_w \hookrightarrow \overline{\mathbb{Q}_p}} \sum_{i=1}^j \lambda_{\tau, n+1-i},$$

then  $\rho|_{G_{F_w}}$  is reducible.

Note that by the proof below (or in applications, by Corollary 4.4.4) we know that  $\text{val}_p(t_j)$  is always at least as big as the right-hand side.

*Proof.* The statement about the characteristic polynomial of  $\varphi^{[F_w^0 : \mathbb{Q}_p]}$  follows from Theorem 7.2.1 above, and Corollary 3.1.2 of [CHT08].

By the usual Baire category argument, we may assume that  $\rho$  is valued in  $\text{GL}_n(E)$  for some finite extension  $E/\mathbb{Q}_p$ . Assume that  $E$  has been chosen large enough that it contains the images of all embeddings  $F_w \hookrightarrow \overline{\mathbb{Q}_p}$  and

that it contains all the roots of  $X^n + \dots + (-1)^j q_w^{j(j-1)/2} t_j X^{n-j} + \dots + (-1)^n q_w^{n(n-1)/2} t_n$ . Assume that for some  $1 \leq j \leq n-1$  we have

$$\mathrm{val}_p(t_j) = \frac{1}{[F_w : F_w^0]} \sum_{\tau: F_w \hookrightarrow \overline{\mathbb{Q}_p}} \sum_{i=1}^j \lambda_{\tau, n+1-i}.$$

Let  $D = D_{\mathrm{cris}}(\rho)$ , and let  $\alpha_1, \dots, \alpha_n \in E$  denote the roots of

$$X^n + \dots + (-1)^j q_w^{j(j-1)/2} t_j X^{n-j} + \dots + (-1)^n q_w^{n(n-1)/2} t_n$$

ordered so that  $\mathrm{val}_p(\alpha_1) \leq \dots \leq \mathrm{val}_p(\alpha_n)$ . For each embedding  $\sigma : F_w^0 \hookrightarrow \overline{\mathbb{Q}_p}$  choose an  $E$ -subspace  $D'_\sigma$  of  $D_\sigma$  such that the eigenvalues of  $\varphi^{[F_w^0 : \mathbb{Q}_p]}$  on it are  $\alpha_1, \dots, \alpha_j$ . Let  $D' = \prod_\sigma D'_\sigma$ , so that  $D'$  is a free  $F_w^0 \otimes_{\mathbb{Q}_p} E$ -submodule of  $D$  of rank  $j$ . We can ensure that  $D'$  is  $\varphi$ -stable by first picking  $D'_{\sigma_0}$  for some fixed embedding  $\sigma_0$  and then defining  $D'_{\sigma \circ (\phi^0)^{-i}} := \varphi^i(D'_{\sigma_0})$  for all  $i$ . Then we have

$$\begin{aligned} t_N(D') &= \frac{\dim_{\mathbb{Q}_p} E}{\dim_{\mathbb{Q}_p} F_w^0} \sum_{i=1}^j \mathrm{val}_p(\alpha_i) \\ &\leq \frac{\dim_{\mathbb{Q}_p} E}{\dim_{\mathbb{Q}_p} F_w^0} \left( \frac{j(j-1)}{2} [F_w^0 : \mathbb{Q}_p] + \mathrm{val}_p(t_j) \right). \end{aligned}$$

Since for each embedding  $\tau : F_w \hookrightarrow E$  the elements of  $\mathrm{HT}_\tau(\rho)$  are  $\lambda_{\tau, n} < \lambda_{\tau, n-1} + 1 < \dots < \lambda_{\tau, 1} + n - 1$ , we also have that

$$\begin{aligned} t_H(D') &\geq \frac{\dim_{\mathbb{Q}_p} E}{\dim_{\mathbb{Q}_p} F_w} \sum_{\tau: F_w \hookrightarrow \overline{\mathbb{Q}_p}} \sum_{i=1}^j (\lambda_{\tau, n+1-i} + i - 1) \\ &= \frac{\dim_{\mathbb{Q}_p} E}{\dim_{\mathbb{Q}_p} F_w^0} [F_w : F_w^0]^{-1} \left( \frac{j(j-1)}{2} [F_w : \mathbb{Q}_p] + [F_w : F_w^0] \mathrm{val}_p(t_j) \right) \\ &\geq t_N(D'). \end{aligned}$$

Since  $D$  is weakly admissible we have  $t_N(D') \geq t_H(D')$ , so we have  $t_H(D') = t_N(D')$ , so that  $\rho$  is reducible.  $\square$

**Corollary 4.5.3.** *Assume that  $\bar{r}|_{G_{F_w}}$  is irreducible, and assume that Axioms  $\tilde{\mathbf{A}}1$  and  $\tilde{\mathbf{A}}2$  hold. Then Axiom  $\mathbf{A}1$  holds.*

*Proof.* Let  $V = F_a$  be a weight of  $\mathrm{GL}_n(k_w)$  such that  $S(V)_{\mathfrak{m}} \neq 0$ , and let  $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F_w, \overline{\mathbb{Q}_p})}$  be the lift of  $a$ . As  $S(F_a) \hookrightarrow \tilde{S}(W_\lambda) \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p}$ , Axiom  $\tilde{\mathbf{A}}1$  implies that  $S(F_a)$  is finite-dimensional. For  $1 \leq j \leq n-1$  let us write  $\tilde{T}_j := (\prod_\tau \tau(\varpi)^{-\sum_{i=1}^j \lambda_{\tau, n+1-i}}) T_j$ . Suppose that  $\alpha_j \in \overline{\mathbb{F}_p}$  for  $1 \leq j \leq n-1$  are eigenvalues of the  $\tilde{T}_j$  on  $(\tilde{S}(W_\lambda) \otimes_{\overline{\mathbb{Z}_p}} \overline{\mathbb{F}_p})_{\mathfrak{m}}$ . By Corollary 4.4.4 it suffices to show that  $\alpha_j = 0$  for all  $j$ . We apply Lemma 4.5.1 with  $M = \tilde{S}(W_\lambda)$ ,  $A = \mathbb{T}[\tilde{T}_1, \dots, \tilde{T}_{n-1}]$ , and the maximal ideal that is the kernel of  $\bar{\theta} : A \rightarrow \mathbb{T}/\mathfrak{m} = \overline{\mathbb{F}_p}$  sending  $\tilde{T}_j$  to  $\alpha_j$ . The  $\theta$ -eigenspace provided by that lemma lies

in  $(\tilde{S}(W_\lambda) \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p)_m$  (as  $\ker \theta \subset \ker \bar{\theta}$ ) and  $\theta(\tilde{T}_j) \in \overline{\mathbb{Z}}_p$  lifts  $\alpha_j$  for all  $j$ . If we had  $\alpha_j \neq 0$  for some  $j$ , it would follow from Axioms  $\tilde{A}1$ ,  $\tilde{A}2$  and Proposition 4.5.2 that there is a lift  $\rho$  of  $\bar{r}|_{G_{F_w}}$  such that  $\rho$  is reducible. Since  $\bar{r}|_{G_{F_w}}$  is irreducible by assumption, this is impossible. The result follows.  $\square$

## 5. WEIGHT ELIMINATION

We preserve all the notation of the preceding section, and assume that Axiom A2 holds. We will however specialise to the case when  $n = 3$  and our fixed place  $w|p$  of  $F$  is split, that is,  $F_w = \mathbb{Q}_p$ . The Serre-type conjecture in [Her09] lets us describe a set  $W_w^?(r)$  of weights in which  $r$  should be modular, and which is conjectured to coincide with the set of all “regular” weights of  $r$ . Our goal in this section is to prove “weight elimination”, i.e. that if  $r$  is modular of some sufficiently generic weight, then this weight does in fact lie in the set  $W_w^?(r)$ .

The construction of  $W_w^?(r)$  is recalled in Subsection 5.1, while elimination itself is proved in Subsection 5.2.

**5.1. The definition of  $W_w^?(r)$ .** The set  $W_w^?(r)$  has a purely local definition depending only on  $\bar{r}|_{G_{F_w}}$ , and we will sometimes refer to it as  $W_w^?(r|_{G_{F_w}})$  in order to emphasise the local nature of the definition. In Section 6 of [Her09] we naturally associated to a tame Galois representation  $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_p)$  a representation  $V(\bar{\rho})$  of  $\mathrm{GL}_3(\mathbb{F}_p)$  over  $\overline{\mathbb{Q}}_p$ . (It is a Deligne–Lusztig representation up to sign and only depends on  $\bar{\rho}|_{I_{\mathbb{Q}_p}}$ .) We also defined a certain operator  $\mathcal{R}$  on the set of irreducible  $\overline{\mathbb{F}}_p$ -representations of  $\mathrm{GL}_3(\mathbb{F}_p)$ . Let  $\mathrm{JH}(\overline{V(\bar{\rho})})$  denote the set of Jordan–Hölder constituents of the reduction mod  $p$  of a lattice inside  $V(\bar{\rho})$ . We let

$$W_w^?(r|_{G_{F_w}}) = \mathcal{R}(\mathrm{JH}(\overline{V(\bar{r}|_{G_{F_w}})})).$$

**5.2. Elimination.** Before we get to our main result on weight elimination we need some preliminary lemmas. We recall from §6.4 in [Her09] the shorthand notation for inertial Galois representations  $\tau : I_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$  that can be extended to  $G_{\mathbb{Q}_p}$ , at least in the special case we need. Suppose that  $\xi \in S_3$  is a permutation of order 3 and that  $\mu \in \mathbb{Z}^3$ . Then we define  $\tau(\xi, \mu) := \psi \oplus \psi^p \oplus \psi^{p^2}$ , where

$$\psi = \omega_3^{\sum_{i=0}^2 \mu_{\xi^i(1)} p^i}.$$

(Recall that the tame fundamental character  $\omega_3$  was defined in Subsection 3.3.)

**Lemma 5.2.1.** *Suppose that  $0 \leq x - y \leq p - 3$  and  $0 \leq y - z \leq p - 3$  and that  $\bar{\rho} : G_{F_w} \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_p)$  is irreducible. Then  $F(x, y, z) \in W_w^?(r)$  if and only if*

$$\bar{\rho}|_{I_{F_w}} \cong \tau(\xi, (x + 2, y + 1, z)), \text{ for some } \xi \in S_3 \text{ of order } 3,$$

or  $x - z > p - 2$  and

$$\bar{\rho}|_{I_{F_w}} \cong \tau(\xi, (z + p, y + 1, x - p + 2)), \text{ for some } \xi \in S_3 \text{ of order 3.}$$

*Proof.* This follows from Proposition 7.4 of [Her09].  $\square$

We need a lemma to help us distinguish niveau 3 inertial Galois representations of the form  $\tau(\xi, (a, b, c))$  with  $\xi \in S_3$  of order 3.

**Lemma 5.2.2.** *Suppose  $n \in \mathbb{Z}$ . Then  $n$  is not divisible by  $p^2 + p + 1$  if and only if one of the following happens:*

- (i)  $n = x + py + p^2z$  with  $x > y \geq z$  and  $x - z \leq p$ .
- (ii)  $n = p^2x + py + z$  with  $x \geq y > z$  and  $x - z \leq p$ .

*The two cases are disjoint. In either case,  $(x, y, z)$  is uniquely determined by  $n$ .*

*Proof.* Without loss of generality, we can assume that  $0 \leq n < p^2 + p + 1$ . (Adding  $p^2 + p + 1$  to  $n$  corresponds to adding 1 to each of  $x, y, z$ .)

If  $n \neq 0$ , in base  $p + 1$  we have  $n - 1 = \alpha(p + 1) + \beta$  with  $0 \leq \alpha \leq p - 1$ ,  $0 \leq \beta \leq p$ . If  $\alpha + \beta \leq p - 1$ , we take  $(x, y, z) = (\alpha + \beta + 1, \alpha, 0)$ . If  $\alpha + \beta \geq p$ , we take  $(x, y, z) = (1, \alpha + 2 - p, \alpha + \beta + 1 - 2p)$ . This establishes the ‘‘only if’’ direction.

Now note that  $0 \leq n < p^2 + p + 1$  is equivalent to  $z = 0$  if  $n$  is as in (i) (resp. to  $x = 1$  if  $n$  is as in (ii)). Each of these cases happens  $\binom{p+1}{2} = (p^2 + p)/2$  times. This implies the remaining claims.  $\square$

**Lemma 5.2.3.** *Suppose  $a > b > c$ ,  $a - c \leq p$ , suppose that  $x > y > z$ ,  $x - z \leq p$ , and suppose that  $a + b + c = x + y + z$ . If*

$$\tau(\xi, (a, b, c)) \cong \tau(\xi', (x, y, z))$$

*for some  $\xi, \xi' \in S_3$  of order 3, then  $\xi = \xi'$  and  $(a, b, c) = (x, y, z)$ .*

*Proof.* Note that  $\tau(\xi, (a, b, c))^\vee \cong \tau(\xi^{-1}, (-c, -b, -a))$ . Thus, by dualising if necessary, we may assume that  $\xi = (1 \ 2 \ 3)$ . It follows that one of  $p^2x + py + z$ ,  $x + py + p^2z$  has to coincide with one of

$$\begin{aligned} a + pb + p^2c, \\ b + pc + p^2a &\equiv (b + 1) + pc + p^2(a - p), \\ c + pa + p^2b &\equiv (c + p) + p(a - 1) + p^2b \end{aligned}$$

modulo  $p^3 - 1$ . Our conditions on  $(a, b, c)$  and  $(x, y, z)$  imply that the first of these five expressions is as in case (ii) and the others are as in case (i) of Lemma 5.2.2. As those two cases are disjoint, we see that  $\xi' = (1 \ 2 \ 3)$ . Lemma 5.2.2 shows that  $(x, y, z)$  has to agree with one of  $(a, b, c)$ ,  $(b + 1, c, a - p)$ ,  $(c + p, a - 1, b)$  modulo  $(p - 1, p - 1, p - 1)\mathbb{Z}$ . Since  $a + b + c = x + y + z$  it follows that  $(a, b, c) = (x, y, z)$ .  $\square$

**Lemma 5.2.4.** *If  $(x, y, z)$  and  $(a, b, c)$  in  $\mathbb{Z}_+^3$  are restricted weights, then  $F(x, y, z) \cong F(a, b, c)$  if and only if  $x - y = a - b$ ,  $y - z = b - c$ , and  $x + y + z \equiv a + b + c \pmod{3(p - 1)}$ .*

*Proof.* This follows immediately from the discussion in §4.1.2.  $\square$

We can now state and prove our main result on weight elimination. (Recall that if  $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_p)$  is continuous and irreducible, then we let  $W_w(\bar{r})$  denote the set of weights  $V$  for which  $\bar{r}$  is modular of weight  $V$ .)

**Theorem 5.2.5.** *Suppose that  $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ , and let  $w|p$  be a place of  $F$  such that  $\bar{r}|_{G_{F_w}}$  is irreducible and such that Axiom A2 is satisfied. Continue to assume that  $F_w = \mathbb{Q}_p$ . Suppose that the weight  $F(x, y, z)$  is such that either*

$$(5.2.6) \quad x - z < p - 3$$

or that

$$(5.2.7) \quad x - y < p - 5, \quad y - z < p - 5, \quad \text{and} \quad x - z > p + 1.$$

Then  $F(x, y, z) \in W_w(\bar{r})$  implies  $F(x, y, z) \in W_w^?(\bar{r}|_{G_{F_w}})$ .

*Proof.* Suppose that  $F(x, y, z) \in W_w(\bar{r})$ .

We first suppose that (5.2.7) holds. Let  $a = y$ ,  $b = x - (p - 1)$ ,  $c = z$ . By Lemma 2.1.3 and Frobenius reciprocity we see that  $F(x, y, z)$  is a constituent of the reduction mod  $p$  of the principal series  $\mathrm{Ind}_{B_3(k_w)}^{\mathrm{GL}_3(k_w)}(\chi^a \otimes \chi^b \otimes \chi^c)$ , where  $B_3 \subset \mathrm{GL}_3$  is the Borel subgroup of upper-triangular matrices and  $\chi : k_w^\times = \mathbb{F}_p^\times \rightarrow \overline{\mathbb{Q}}_p^\times$  is the Teichmüller lift. (Note that the principal series is unchanged, up to semisimplification, when the characters are permuted.) By Axiom A2 there is a lift of  $\bar{r}|_{G_{F_w}}$  to a potentially semistable representation  $\rho : G_{F_w} \rightarrow \mathrm{GL}_3(\overline{\mathbb{Q}}_p)$  with Hodge–Tate weights  $-2, -1, 0$  and  $\mathrm{WD}(\rho)|_{I_{F_w}} \cong \tilde{\omega}^a \oplus \tilde{\omega}^b \oplus \tilde{\omega}^c$ . By Theorem 3.3.13, we have  $\bar{r}|_{I_{F_w}} \cong \psi \oplus \psi^p \oplus \psi^{p^2}$ , where either

$$(5.2.8) \quad \psi = \omega_3^{(y+a_0)+p(z+a_2)+p^2(x-p+1+a_1)}$$

or

$$(5.2.9) \quad \psi = \omega_3^{(y+2-a_2)+p(x-p+3-a_1)+p^2(z+2-a_0)}$$

with  $(a_0, a_1, a_2) \in \{(1, 1, 1), (1, 2, 0), (2, 1, 0)\}$ . (Here we use that  $x - y < p - 3$ ,  $y - z < p - 3$ ,  $x - z > p + 1$ .)

Next we use that  $F(x, y, z)$  is a constituent of the reduction of two cuspidal representations. Choose a  $\mathbb{Q}_p$ -linear embedding of  $F_{w,3}$  into  $\overline{\mathbb{Q}}_p$  and let  $\chi_3 : k_{w,3}^\times \rightarrow F_{w,3}^\times \rightarrow \overline{\mathbb{Q}}_p^\times$  denote the Teichmüller lift. By formula (7.8) in [Her09] we see that  $F(x, y, z)$  is a constituent of the reduction mod  $p$  of the cuspidal representation  $R_{\mathbb{T}}^\theta$  in the notation of §2.4, where  $\theta = \chi_3^{a+pb+p^2c}$  and  $a = y + 1$ ,  $b = x - (p - 1)$ ,  $c = z - 1$ . (Note that all highest weights in that formula are restricted, due to our bounds on  $(x, y, z)$ .) By Axiom A2 there is a lift of  $\bar{r}|_{G_{F_w}}$  to a potentially semistable representation  $\rho : G_{F_w} \rightarrow \mathrm{GL}_3(\overline{\mathbb{Q}}_p)$  with Hodge–Tate weights  $-2, -1, 0$  and  $\mathrm{WD}(\rho)|_{I_{F_w}} \cong \tilde{\omega}_3^{a+pb+p^2c} \oplus \tilde{\omega}_3^{b+pc+p^2a} \oplus$

$\tilde{\omega}_3^{c+pa+p^2b}$ . By Theorem 3.3.13, we have  $\bar{r}|_{I_{F_w}} \cong \psi \oplus \psi^p \oplus \psi^{p^2}$ , where either

$$(5.2.10) \quad \psi = \omega_3^{(y+1+a_0)+p(z-1+a_2)+p^2(x-p+1+a_1)}$$

with  $(a_0, a_1, a_2) \in \{(0, 2, 1), (1, 1, 1), (1, 2, 0)\}$  or

$$(5.2.11) \quad \psi = \omega_3^{(y+1+a_0)+p(x-p+1+a_2)+p^2(z-1+a_1)}$$

with  $a_0, a_1, a_2 \in [0, 2]$  with  $a_1 + a_2 + a_3 = 3$ . (Here we use that  $x - y < p - 2$ ,  $y - z < p - 5$ ,  $x - z > p$ .)

Similarly we claim that  $F(x, y, z)$  is a constituent of the reduction mod  $p$  of the cuspidal representation  $R_{\mathbb{T}}^{\theta}$  where  $\theta = \chi_3^{c+pb+p^2a}$  and  $a = x + 1$ ,  $b = z + p - 1$ ,  $c = y - 1$ . Equivalently,  $F(x, y, z)^{\vee} \cong F(-z, -y, -x)$  is a constituent of the reduction of  $(R_{\mathbb{T}}^{\theta})^{\vee} \cong R_{\mathbb{T}}^{\theta^{-1}}$ . (For the last isomorphism see p. 136 in [DL76].) This is again true by formula (7.8) in [Her09]. By Axiom A2 there is a lift of  $\bar{r}|_{G_{F_w}}$  to a potentially semistable representation  $\rho : G_{F_w} \rightarrow \mathrm{GL}_3(\overline{\mathbb{Q}}_p)$  with Hodge–Tate weights  $-2, -1, 0$  and  $\mathrm{WD}(\rho)|_{I_{F_w}} \cong \tilde{\omega}_3^{c+pb+p^2a} \oplus \tilde{\omega}_3^{b+pa+p^2c} \oplus \tilde{\omega}_3^{a+pc+p^2b}$ . By Theorem 3.3.13, we have  $\bar{r}|_{I_{F_w}} \cong \psi \oplus \psi^p \oplus \psi^{p^2}$ , where either

$$(5.2.12) \quad \psi = \omega_3^{(y+2-a_0)+p(z+1-a_2)+p^2(x-p+4-a_1)}$$

with  $a_0, a_1, a_2 \in [0, 2]$  with  $a_1 + a_2 + a_3 = 3$ , or

$$(5.2.13) \quad \psi = \omega_3^{(y+2-a_0)+p(x-p+3-a_2)+p^2(z+2-a_1)}$$

with  $(a_0, a_1, a_2) \in \{(0, 2, 1), (1, 1, 1), (1, 2, 0)\}$ . (Here we use that  $x - y < p - 5$ ,  $y - z < p - 2$ ,  $x - z > p$ .)

Lemma 5.2.3 allows us to distinguish between the various inertial Galois representations. (Note that whenever Theorem 3.3.13 applies, it shows that  $\bar{r}|_{I_{F_w}} \cong \tau(\xi, (a + b_0, b + b_1, c + b_2))$  for some  $\xi \in S_3$  of order 3 and some  $b_i \in [0, 2]$  with  $\sum b_i = 3$ . We have  $a + b_0 > b + b_1 > c + b_2$  and  $(a + b_0) - (c + b_2) \leq p$  due to the assumptions placed on  $(a, b, c)$ . Moreover, whenever we applied that theorem, we had  $a + b + c = x + y + z + p - 1$ .) In particular, we see that the possibilities for  $\bar{r}|_{I_{F_w}}$  described in (5.2.8), (5.2.10), (5.2.12) are distinct from the possibilities described in (5.2.9), (5.2.11), (5.2.13).

The above results thus show that either

$$(5.2.14) \quad \bar{r}|_{I_{F_w}} \cong \tau((1 \ 3 \ 2), (y + b_0, x - p + 1 + b_1, z + b_2))$$

with  $(b_0, b_1, b_2) \in \{(1, 2, 0), (2, 1, 0)\}$  (use (5.2.8) and (5.2.10)), or

$$(5.2.15) \quad \bar{r}|_{I_{F_w}} \cong \tau((1 \ 2 \ 3), (y + b_0, x - p + 1 + b_1, z + b_2))$$

with  $(b_0, b_1, b_2) \in \{(1, 1, 1), (2, 1, 0)\}$  (use (5.2.9) and (5.2.13)).

On the other hand, Lemma 5.2.1 shows that  $F(x, y, z) \in W_w^?(\bar{r}|_{G_{F_w}})$  is equivalent to  $\bar{r}|_{I_{F_w}}$  being isomorphic to  $\tau(\xi, (x + 2, y + 1, z))$  or  $\tau(\xi, (z + p, y + 1, x - p + 2))$  for some  $\xi \in S_3$  of order 3, but it is not hard to see that this is equivalent to (5.2.14) or (5.2.15). It follows that  $F(x, y, z) \in W_w^?(\bar{r}|_{G_{F_w}})$ .

Now suppose that (5.2.6) holds. By Axiom A2 there is a lift of  $\bar{r}|_{G_{F_w}}$  to a crystalline representation  $\rho : G_{F_w} \rightarrow \mathrm{GL}_3(\overline{\mathbb{Q}_p})$  with Hodge–Tate weights  $-(x+2)$ ,  $-(y+1)$ ,  $-z$ . Since  $\bar{r}|_{G_{F_w}}$  is irreducible and  $(x+2) - z \leq p-2$ , a standard calculation in Fontaine–Laffaille theory (cf. Theorem 1.2' of [Sch08c]) shows that  $\bar{r}|_{I_{F_w}} \cong \tau(\xi, (x+2, y+1, z))$  for some  $\xi \in S_3$  of order 3 and thus  $F(x, y, z) \in W_w^?(\bar{r}|_{G_{F_w}})$  by Lemma 5.2.1. (Alternatively we could proceed as in the previous case, using for example the two principal series representations whose reduction contains  $F(x, y, z)$ . But then we would need to assume that  $x - y > 2$ ,  $y - z > 2$ , and  $x - z < p - 5$ .)  $\square$

## 6. WEIGHT CYCLING FOR $\mathrm{GL}_3$

In this section we further develop the general weight cycling formalism of Subsection 2.3 in the particular case of the group  $\mathrm{GL}_3(\mathbb{Q}_p)$ , and then use it to prove our main theorem. More precisely, in Subsection 6.1 we prove Proposition 6.1.3, which makes Corollary 2.3.3 completely explicit in this context, while in Subsection 6.2, we prove Theorem 6.2.3, which is the main theorem of the paper in the axiomatic setting of Section 4; as explained in the introduction, it establishes that if  $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}_p})$  is modular of some strongly generic weight, then the set  $W_w^?(\bar{r})$  of conjectured weights consists precisely of those generic weights for which  $\bar{r}$  is modular. (See Definition 6.2.2 for the definition of generic and strongly generic weights.)

**6.1. Local results.** We start with a basic but important lemma on parabolically induced representations for the group  $\mathrm{GL}_3(\mathbb{F}_p)$ . We introduce the following notation for standard parabolic subgroups of  $\mathrm{GL}_3$ : let  $P_1 := P_{(0,0,-1)} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & & * \end{pmatrix}$  and  $P_2 := P_{(0,-1,-1)} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & & * \end{pmatrix}$ .

Suppose that  $V$  is an irreducible  $\overline{\mathbb{F}_p}$ -representation of  $\mathrm{GL}_3(\mathbb{F}_p)$ . We say that  $V$  lies in the lower alcove if  $V \cong F(x, y, z)$  with  $x - y \geq 0$ ,  $y - z \geq 0$ , and  $x - z < p - 2$ . We say that  $V$  lies in the closure of the lower alcove if the same conditions hold, except that  $x - z \leq p - 2$ . We say that  $V$  lies in the upper alcove if  $V \cong F(x, y, z)$  with  $x - y < p - 1$ ,  $y - z < p - 1$ , and  $x - z > p - 2$ . (These notions do not depend on the choice of  $(x, y, z)$  by Lemma 5.2.4.)

**Lemma 6.1.1.** *Suppose that  $(a, b, c) \in \mathbb{Z}^3$  with  $a - b > 0$ ,  $b - c > 0$ , and  $a - c < p - 1$ . Then the induced representation  $\mathrm{Ind}_{P_2(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(F(a) \otimes F(b, c))$  is of length three with irreducible constituents given by*

$$F(b, c, a - p + 1), F(b + p - 1, a, c), F(a, b, c).$$

*The induced representation  $\mathrm{Ind}_{P_2(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)}(F(a) \otimes F(c, b - p + 1))$  is of length six with irreducible constituents given by*

$$F(c + p - 1, b, a - p + 1), F(c + p - 1, a, b), F(c + p - 2, a, b + 1), \\ F(a - 1, b, c + 1), F(b - 1, c, a - p + 2), F(a, c, b - p + 1).$$

These nine irreducible representations of  $\mathrm{GL}_3(\mathbb{F}_p)$  are non-isomorphic, by Lemma 5.2.4. (It helps to note that the second, the fourth, and the ninth weight lie in the upper alcove, and the others lie in the closure of the lower alcove.)

*Proof.* Recall that we denote by  $B_n \subset \mathrm{GL}_n$  the Borel subgroup of upper-triangular matrices and by  $T_n \subset B_n$  the diagonal maximal torus. We have a short exact sequence

$$0 \rightarrow F(c, b - p + 1) \rightarrow \mathrm{Ind}_{B_2(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} (F(b) \otimes F(c)) \rightarrow F(b, c) \rightarrow 0.$$

(We get the two maps using Frobenius reciprocity together with Lemma 2.3 in [Her11a]. For dimension reasons exactness follows, since the representations at the ends are non-isomorphic.) Tensoring with  $F(a)$  and parabolically inducing to  $\mathrm{GL}_3(\mathbb{F}_p)$  we see that the principal series representation

$$\Pi := \mathrm{Ind}_{B_3(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(a) \otimes F(b) \otimes F(c))$$

is an extension between the two induced representations in the statement of this lemma. But  $\Pi = \overline{R}_1(a, b, c)$  in the Grothendieck group, for example by Lemma 4.7 in [Her09] (note that a factor  $(-1)^{n-r}$  is missing on the left-hand side). Here,  $R_\xi(\mu)$  for  $\xi \in S_3$  and  $\mu \in X^*(T_3)$  denotes a Deligne–Lusztig representation of  $\mathrm{GL}_3(\mathbb{F}_p)$  over  $\overline{\mathbb{Q}}_p$ . We can thus determine the irreducible constituents of  $\Pi$  from the formula in the proof of Proposition 7.4 in [Her09]. Decomposing dual Weyl modules using Proposition 3.18 in [Her09], we obtain the nine irreducible representations listed above. Since they are distinct, for any irreducible constituent  $V$  of  $\Pi$ , the principal series  $\Pi$  has a unique quotient with socle  $V$ . We compute the socle of  $\mathrm{Ind}_{P_2(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(a) \otimes F(b, c))$  using Lemma 2.3 in [Her11a]: if  $V = F(x, y, z)$  is in the socle, then by Frobenius reciprocity we have a  $\mathrm{GL}_1(\mathbb{F}_p) \times \mathrm{GL}_2(\mathbb{F}_p)$ -linear map  $F(z) \otimes F(x, y) \rightarrow F(a) \otimes F(b, c)$ , so by our bounds on  $(a, b, c)$  we see that  $V \cong F(b, c, a - p + 1)$ . Thus  $\mathrm{Ind}_{P_2(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(a) \otimes F(b, c))$  is the unique quotient of  $\Pi$  with socle  $F(b, c, a - p + 1)$ .

Let  $\alpha_2$  be the simple root  $(0, 1, -1) \in X^*(T_3)$ , whose corresponding simple reflection is  $(2\ 3)$  under the natural identification of the Weyl group with  $S_3$ . Jantzen [Jan84] considers a natural intertwining map

$$\overline{T}_{\alpha_2} : \mathrm{Ind}_{B_3(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(a) \otimes F(b) \otimes F(c)) \rightarrow \mathrm{Ind}_{B_3(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(a) \otimes F(c) \otimes F(b))$$

and gives in Satz 4.1 the following formula for its image, in the Grothendieck group of  $\mathrm{GL}_3(\mathbb{F}_p)$ -representations over  $\overline{\mathbb{F}}_p$ :

$$\mathrm{im}(\overline{T}_{\alpha_2}) = \frac{1}{2} \sum_{0 \leq l \leq b-c} (\overline{R}_1(a, b-l, c+l) - \overline{R}_{(2\ 3)}(a, b-l, c+l)).$$

Note that Jantzen computes the restriction of this representation to  $\mathrm{SL}_3(\mathbb{F}_p)$  in §5.2 of [Jan84], but it is not completely straightforward to deduce the result for  $\mathrm{GL}_3(\mathbb{F}_p)$  due to the fact that the restriction of  $\Pi$  to  $\mathrm{SL}_3(\mathbb{F}_p)$  may

have repeated Jordan–Hölder factors. Instead, we redo his calculation for  $\mathrm{GL}_3(\mathbb{F}_p)$ , using his result on the reduction mod  $p$  of Deligne–Lusztig representations. From the proof of Proposition 7.4 in [Her09] we find that for all  $(i, j, k) \in \mathbb{Z}^3$ ,  $\overline{R_1(i, j, k)} - \overline{R_{(2\ 3)}(i, j, k)}$  equals

$$\begin{aligned} & (W(j + p - 1, i, k) - W(j + p - 2, i, k + 1)) \\ & \quad + (W(i, j, k) - W(i, j - 1, k + 1)) \\ & + (W(j, k, i - p + 1) - W(j - 1, k + 1, i - p + 1)) \\ & \quad + (W(k + p - 1, i, j) - W(k + p - 2, i, j + 1)), \end{aligned}$$

where  $W(\mu)$  denotes the dual Weyl module of highest weight  $\mu$ . Thus the sum over  $l$  above is telescoping and we obtain eight terms initially. But two of them are dual Weyl modules with non-dominant highest weights, which we can make dominant using formula (3.5) in [Her09]. After simplifying we find that  $\mathrm{im}(\overline{T}_{\alpha_2})$  equals

$$W(b + p - 1, a, c) + W(a, b, c) + W(b, c, a - p + 1) - W(c + p - 2, a, b + 1).$$

By Proposition 3.18 in [Her09] this equals

$$F(b + p - 1, a, c) + F(a, b, c) + F(b, c, a - p + 1).$$

Finally,  $\mathrm{im}(\overline{T}_{\alpha_2})$  is isomorphic to  $\mathrm{Ind}_{P_2(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(a) \otimes F(b, c))$ , since both are quotients of  $\Pi$  with socle  $F(b, c, a - p + 1)$ . (Note that even the codomain of  $\overline{T}_{\alpha_2}$  has this socle.) This completes the proof of the first part of the lemma. The second part follows from our computation of the union of the constituents above.  $\square$

*Remark 6.1.2.* With some more work it is possible to compute the complete submodule structure of  $\mathrm{Ind}_{B_3(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(a) \otimes F(b) \otimes F(c))$ . This uses also Jantzen’s 3-step filtration with semisimple graded pieces of this representation [Jan84].

Now we go back to the setting of Section 2, in the special case of the algebraic group  $G = \mathrm{GL}_3$  with  $F$  taken to be  $\mathbb{Q}_p$ . (We could equally well assume that  $F/\mathbb{Q}_p$  is totally ramified, but we will not need it.) We have  $K = \mathrm{GL}_3(\mathbb{Z}_p)$ . Suppose that  $V$  is a weight of the group  $\mathrm{GL}_3(\mathbb{F}_p)$ , i.e. an irreducible  $\overline{\mathbb{F}}_p$ -representation of  $\mathrm{GL}_3(\mathbb{F}_p)$ . Recall that we defined Hecke operators  $\overline{T}_1 = \overline{T}_{(0,0,1),\varpi}$  and  $\overline{T}_2 = \overline{T}_{(0,1,1),\varpi}$  in  $\mathcal{H}_G(V)$  in Subsection 4.2.

**Proposition 6.1.3.** *Suppose that  $V = F(x, y, z)$ , where  $(x, y, z)$  is a restricted weight and that  $\pi$  is a smooth  $\mathrm{GL}_3(\mathbb{Q}_p)$ -representation over  $\overline{\mathbb{F}}_p$ .*

- (i) *Suppose  $x - y > 0$ ,  $y - z > 0$ , and  $x - z < p - 1$ . If  $\overline{T}_1$  (resp.  $\overline{T}_2$ ) fails to be injective on  $(V \otimes_{\overline{\mathbb{F}}_p} \pi)^K$ , then  $(V' \otimes_{\overline{\mathbb{F}}_p} \pi)^K \neq 0$  where  $V'$  is one of*

$$F(z + p - 1, x, y), \quad F(x, z, y - p + 1);$$

respectively,

$$F(y, z, x - p + 1), F(y + p - 1, x, z).$$

- (ii) Suppose  $x - y < p - 1$ ,  $y - z < p - 1$ , and  $x - z > p - 1$ . If  $\overline{T}_1$  (resp.  $\overline{T}_2$ ) fails to be injective on  $(V \otimes_{\overline{\mathbb{F}}_p} \pi)^K$ , then  $(V' \otimes_{\overline{\mathbb{F}}_p} \pi)^K \neq 0$  where  $V'$  is one of

$$\begin{aligned} &F(x, z + p - 1, y), F(x - 1, z + p - 1, y + 1), \\ &F(y - 1, x - p + 1, z + 1), F(z + p - 2, y, x - p + 2), \\ &F(z + 2p - 2, x, y); \end{aligned}$$

respectively,

$$\begin{aligned} &F(y, x - p + 1, z), F(y - 1, x - p + 1, z + 1), \\ &F(x - 1, z + p - 1, y + 1), F(z + p - 2, y, x - p + 2), \\ &F(y, z, x - 2p + 2). \end{aligned}$$

In each case, the last weight lies in the upper alcove and the other weights lie in the closure of the lower alcove.

*Proof.* We apply Corollary 2.3.3 with  $\lambda = (0, 0, 1)$ , respectively  $\lambda = (0, 1, 1)$ . In these two cases it is clear that  $K \cap {}^{\lambda(\varpi)}K$  contains  $\ker(K \rightarrow \mathrm{GL}_3(\mathbb{F}_p))$ , so by Lemma 2.1.4 we see that  $\pi|_K$  contains an irreducible constituent of

$$\ker(\mathrm{Ind}_{P_{-\lambda}(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} V^{N_{-\lambda}(\mathbb{F}_p)} \rightarrow V).$$

By Lemma 2.3 in [Her11a] the induced representation equals

$$I_1 := \mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(x, y) \otimes F(z)), \text{ resp. } I_2 := \mathrm{Ind}_{P_2(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(x) \otimes F(y, z)).$$

To compute the constituents of  $I_2$ , we apply Lemma 6.1.1 with  $a = x$ ,  $b = y$ ,  $c = z$  if  $x - z < p - 1$  and with  $a = x$ ,  $b = z + p - 1$ ,  $c = y$  if  $x - z > p - 1$ . This completes the case when  $\overline{T}_2$  fails to be injective.

We will compute the constituents of  $I_1$  from those of  $I_2$ . The outer automorphism  $\zeta : g \mapsto \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \cdot {}^t g^{-1} \cdot \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$  of  $\mathrm{GL}_3(\mathbb{F}_p)$  maps every irreducible  $\overline{\mathbb{F}}_p$ -representation  $F(x, y, z)$  to its dual  $F(-z, -y, -x)$ . (Consider formal characters on the level of algebraic group representations.) Thus, when we apply  $\zeta$  to  $I_2$  we obtain  $\mathrm{Ind}_{P_1(\mathbb{F}_p)}^{\mathrm{GL}_3(\mathbb{F}_p)} (F(-z, -y) \otimes F(-x))$  and its constituents are dual to the ones of  $I_2$ . It remains to relabel  $(-z, -y, -x)$  as  $(x, y, z)$  to obtain  $I_1$ . Concretely, we obtain the constituents of  $I_1$  from those of  $I_2$  by first dualising and then relabelling  $(-z, -y, -x)$  as  $(x, y, z)$ . It is now easy to verify that we get the required constituents, so this completes the case when  $\overline{T}_1$  fails to be injective.  $\square$

**6.2. The main theorem.** We preserve all the notation of Section 4, and assume that Axioms A1 and A2 holds. As in Section 5 we assume furthermore that  $n = 3$  and that  $F_w = \mathbb{Q}_p$ . In this subsection we prove our main theorem in the axiomatic setting. However, before doing so, we have to make precise the notion of a weight being (strongly) generic. To this end we introduce the following definitions.

**Definition 6.2.1.** Suppose that  $\delta \in \mathbb{Z}_{\geq 0}$ . We say that an irreducible  $\overline{\mathbb{F}}_p$ -representation of  $\mathrm{GL}_3(\overline{\mathbb{F}}_p)$  is  $\delta$ -generic if it is isomorphic to  $F(x, y, z)$  for some  $(x, y, z) \in \mathbb{Z}_+^3$  such that

$$\begin{aligned} -1 + \delta &< x - y < p - 1 - \delta, \\ -1 + \delta &< y - z < p - 1 - \delta, \\ |x - z - (p - 2)| &> \delta. \end{aligned}$$

Note that by Lemma 5.2.4 this definition does not depend on the choice of  $(x, y, z)$ .

**Definition 6.2.2.** We say that a weight  $V$  is *generic* (resp. *strongly generic*) if  $V$  is 4-generic (resp. 6-generic).

If  $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_3)$  is continuous and irreducible, then we recall that  $W_w(\bar{r})$  denotes the set of weights  $V$  for which  $\bar{r}$  is modular of weight  $V$ , and we let  $W_{\mathrm{gen},w}(\bar{r}) := \{V \in W_w(\bar{r}) : V \text{ is generic}\}$ . With this, we are finally ready to state and prove our main theorem.

**Theorem 6.2.3.** *Suppose that  $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\overline{\mathbb{F}}_p)$  is a continuous representation, and let  $w|p$  be a place of  $F$  such that  $\bar{r}|_{G_{F_w}}$  is irreducible and such that Axioms A1 and A2 are satisfied. Continue to assume that  $F_w = \mathbb{Q}_p$ . If  $\bar{r}$  is modular of some strongly generic weight, then  $W_{\mathrm{gen},w}(\bar{r}) = W_w^?( \bar{r})$ .*

*Proof.* By Theorem 5.2.5 we know that  $W_{\mathrm{gen},w}(\bar{r}) \subset W_w^?( \bar{r})$ . To establish the reverse inclusion we will show that all weights in  $W_w^?( \bar{r})$  are generic and that  $W_w^?( \bar{r}) \subset W_w(\bar{r})$ .

*Step 1:* We show that either (a)  $\bar{r}|_{I_{F_w}}$  or (b)  $\bar{r}|_{I_{F_w}}^{\vee} \otimes \bar{\varepsilon}^2$  is isomorphic to  $\tau((1 \ 2 \ 3), (a + 2, b + 1, c))$  for some  $(a, b, c) \in \mathbb{Z}^3$  such that  $a - b > 5$ ,  $b - c > 4$ ,  $a - c < p - 7$ . By assumption we can pick a 6-generic weight  $F(x, y, z) \in W_w(\bar{r})$ . Thus  $F(x, y, z) \in W_w^?( \bar{r})$  and we get from Lemma 5.2.1 that either  $\bar{r}|_{I_{F_w}} \cong \tau(\xi, (x + 2, y + 1, z))$ , or  $x - z > p - 2$  and  $\bar{r}|_{I_{F_w}} \cong \tau(\xi, (z + p, y + 1, x - p + 2))$ , where  $\xi \in S_3$  is of order 3. If  $x - z < p - 2$  and  $\xi = (1 \ 2 \ 3)$  then conclusion (a) holds with  $(a, b, c) = (x, y, z)$ . If  $x - z > p - 2$  and  $\bar{r}|_{I_{F_w}}$  is isomorphic to  $\tau((1 \ 2 \ 3), (z + p, y + 1, x - p + 2))$  or to  $\tau((1 \ 3 \ 2), (x + 2, y + 1, z)) \cong \tau((1 \ 2 \ 3), (z + p + 1, y, x - p + 2))$ , then it is easy to check that conclusion (a) holds. For the three remaining cases, note that either  $\bar{r}|_{G_{I_w}}^{\vee} \otimes \bar{\varepsilon}^2 \cong \tau((1 \ 3)\xi(1 \ 3), (-z + 2, -y + 1, -x))$  or  $x - z > p - 2$  and  $\bar{r}|_{G_{I_w}}^{\vee} \otimes \bar{\varepsilon}^2 \cong \tau((1 \ 3)\xi(1 \ 3), (-x + p, -y + 1, -z - p + 2))$ . Compared to the first three cases,  $(x, y, z)$  is interchanged with  $(-z, -y, -x)$  and  $\xi$  with  $(1 \ 3)\xi(1 \ 3)$ . We thus find that conclusion (b) holds.

*Step 2:* For the next few steps we suppose that conclusion (a) holds in Step 1. We now give an explicit description of the set  $W_w^?(\bar{r}|_{G_{F_w}})$ . (This description follows from Lemma 5.2.1; see Lemma 7.6 in [Her09] for details.) It consists of the “obvious weights in the lower alcove”,

$$F(a, b, c), F(c + p - 2, a, b + 1), F(b, c - 1, a - p + 2),$$

the “obvious weights in the upper alcove”,

$$F(c + p - 2, b + 1, a - p + 1), F(b + p - 1, a + 1, c - 1), F(a, c, b - p + 1),$$

and the “shadow weights in the upper alcove”,

$$F(c + p - 2, b, a - p + 2), F(b + p - 1, a, c), F(a, c - 1, b - p + 2).$$

(In general we say that a weight  $F(x, y, z)$  that lies in the upper alcove is the *shadow* of the weight  $F(z + p - 2, y, x - p + 2)$  in the lower alcove. The two highest weights are related by a simple reflection in the affine Weyl group.) Note that the map  $\theta : (a, b, c) \mapsto (c + p - 2, a, b + 1)$  preserves  $\tau((1\ 2\ 3), (a + 2, b + 1, c))$  and the bounds  $a - b > 5$ ,  $b - c > 4$ ,  $a - c < p - 7$ . In particular, it induces a permutation of  $W_w^?(\bar{r}|_{G_{F_w}})$ . It cyclically permutes each of the three kinds of weights above. (Note that it preserves these weights, though not the particular choice of highest weights used in the list above. See Lemma 5.2.4.)

We now verify that all nine weights above are 4-generic. For  $F(a, b, c)$  we have

$$\begin{aligned} a - b &= (a - b - 5) + 5 > 3, \\ b - c &= (b - c - 4) + 4 > 3, \\ a - c &= p - 7 - (p - 7 - a + c) < p - 6, \end{aligned}$$

and this also covers its shadow  $F(c + p - 2, b, a - p + 2)$ . For  $F(c + p - 2, b + 1, a - p + 1)$  we have

$$\begin{aligned} (c + p - 2) - (b + 1) &= p - 7 - (b - c - 4) < p - 5, \\ (b + 1) - (a - p + 1) &= p - 5 - (a - b - 5) < p - 5, \\ (c + p - 2) - (a - p + 1) &= (p - 7 - a + c) + p + 4 > p + 2. \end{aligned}$$

We have written the inequalities in such a way that they immediately carry over for the other six weights using  $\theta$ . (Note that  $\theta$  permutes  $\{a - b - 5, b - c - 4, p - 7 - a + c\}$ .)

Next we consider an arbitrary element  $V \in W_w(\bar{r})$ .

*Step 3:* Suppose that  $V$  is an obvious weight in the lower alcove. Using  $\theta$ , we may assume without loss of generality that  $V \cong F(a, b, c)$ . We have an  $\mathcal{H}_G(V)$ -equivariant isomorphism

$$S(V)_\mathfrak{m} \cong (V \otimes_{\mathbb{F}_p} S_\mathfrak{m})^{\mathrm{GL}_3(\mathcal{O}_{F_w})}.$$

Axiom A1 implies that both  $\bar{T}_1$  and  $\bar{T}_2$  in  $\mathcal{H}_G(V)$  act nilpotently on the finite-dimensional vector space  $S(V)_\mathfrak{m}$ . We can thus apply weight cycling for both Hecke operators.

In the case of  $\bar{T}_1$  Proposition 6.1.3 shows that  $S(V')_{\mathfrak{m}} \neq 0$  where  $V'$  is isomorphic to either  $F(c+p-1, a, b)$  or  $F(a, c, b-p+1)$ . Suppose the first possibility holds. Then  $F(c+p-1, a, b) \in W_w(\bar{r}) \subset W_w^?(\bar{r}|_{G_{F_w}})$  by Theorem 5.2.5. Since  $(c+p-1) - b < p-2$  it would have to equal one of the obvious weights in the lower alcove. But Lemma 5.2.4 implies that this is not the case. Thus  $F(a, c, b-p+1) \in W_w(\bar{r})$ .

The case of  $\bar{T}_2$  is very similar. We eliminate  $F(b, c, a-p+1)$  and deduce that  $F(b+p-1, a, c) \in W_w(\bar{r})$ .

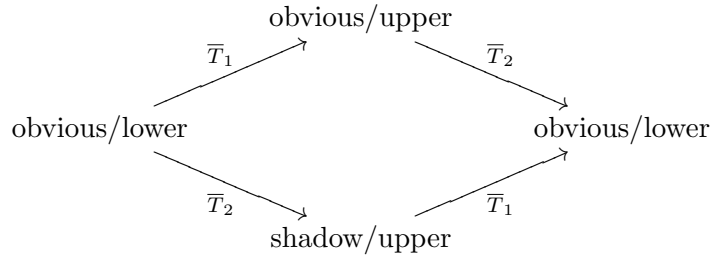
*Step 4:* Suppose that  $V$  is an obvious weight in the upper alcove. Using  $\theta$ , we may assume without loss of generality that  $V \cong F(a, c, b-p+1)$ . We apply weight cycling with  $\bar{T}_2$  and deduce from Proposition 6.1.3 that  $S(V')_{\mathfrak{m}} \neq 0$  where  $V'$  is isomorphic to one of  $F(c+p-1, a, b)$ ,  $F(c+p-2, a, b+1)$ ,  $F(a-1, b, c+1)$ ,  $F(b-1, c, a-p+2)$ , and  $F(c+p-1, b, a-p+1)$ . As in the previous step we can eliminate all these possibilities except for the second. Let us give some detail for the last weight, the only one that lies in the upper alcove. We have

$$\begin{aligned} (c+p-1) - b &= p-5 - (b-c-4) < p-5, \\ b - (a-p+1) &= p-6 - (a-b-5) < p-5, \\ (c+p-1) - (a-p+1) &= (p-7-a+c) + p+5 > p+1, \end{aligned}$$

so we may indeed apply Theorem 5.2.5 to conclude that  $F(c+p-1, b, a-p+1) \in W_w^?(\bar{r}|_{G_{F_w}})$ , a contradiction. Thus  $F(c+p-2, a, b+1) \in W_w(\bar{r})$ .

*Step 5:* Suppose that  $V$  is a shadow weight in the upper alcove. Using  $\theta$ , we may assume without loss of generality that  $V \cong F(b+p-1, a, c)$ . We apply weight cycling with  $\bar{T}_1$  and deduce that  $S(V')_{\mathfrak{m}} \neq 0$  where  $V'$  is isomorphic to one of  $F(b, c, a-p+1)$ ,  $F(b-1, c, a-p+2)$ ,  $F(a-1, b, c+1)$ ,  $F(c+p-2, a, b+1)$ , and  $F(c+p-1, b, a-p+1)$ . In the previous two steps we eliminated all these possibilities except for the third. So  $F(c+p-2, a, b+1) \in W_w(\bar{r})$ .

*Step 6:* Finally, we put the previous three steps together and take into account the symmetry  $\theta$ . Schematically the cycling process does the following.



It is clear that it passes through all nine weights in  $W_w^?(\bar{r}|_{G_{F_w}})$ . (The weights on the left and on the right are related by  $\theta$ .) This completes the proof of the theorem, in case conclusion (a) in Step 1 holds.

*Step 7:* If conclusion (b) holds in Step 1, the situation is dual to the one we considered: since  $W_w^?(\bar{r}|_{G_{F_w}})^\vee = W_w^?(\bar{r}|_{G_{F_w}}^\vee \otimes \bar{\varepsilon}^2)$  by Proposition 6.23 in [Her09], the weights in Step 2 become dualised. A weight  $V$  lies in the same alcove as its dual  $V^\vee$  and if one of them is generic, so is the other. The argument in the proof of Proposition 6.1.3 shows that for  $i = 1, 2$ , the implied weights for  $V$  if  $\bar{T}_i$  fails to be injective are dual to the implied weights for  $V^\vee$  if  $\bar{T}_{3-i}$  fails to be injective.  $\square$

*Remark 6.2.4.* In Steps 4 and 5 we only applied weight cycling with one of the two Hecke operators. In either situation it is not hard to see that, if we apply weight cycling with the other Hecke operator we can only eliminate two out of the five possibilities.

## 7. GLOBAL APPLICATIONS

In this section we show that the axioms of Section 4 are satisfied for certain spaces of automorphic forms on definite unitary groups, and we establish our main theorems for these groups. Subsection 7.1 introduces notation and terminology related to unitary groups and automorphic forms on them. Subsection 7.2 presents various results relating Galois representations, automorphic forms, and Serre weights; we signal in particular Definition 7.3.5, in which we give precise meaning to the notion of a global residual Galois representation being modular of a given Serre weight. Subsection 7.4 verifies that the axioms of Section 4 are satisfied, and the main theorems are deduced in Subsection 7.5. Finally, Subsection 7.6 uses automorphic induction of Hecke characters to produce specific examples to which our main theorems apply.

**7.1. Automorphic forms on unitary groups.** Throughout this subsection, and those that follow, we fix an imaginary CM field  $F$  with maximal totally real field subfield  $F^+$ , and let  $c$  denote the non-trivial element of  $\text{Gal}(F/F^+)$ . We assume that every place of  $F^+$  above  $p$  splits in  $F$ . (Note that in our main theorems, we will further assume that  $p$  splits completely in  $F$ .) We now proceed to introduce the various objects that will be required for our global theorems.

**7.1.1. Unitary groups.** We fix a positive integer  $n$  (which will eventually be specialised to equal 3), and let  $G$  be a reductive group over  $F^+$  which is an outer form of  $\text{GL}_n$ , which splits over  $F$ , and which has the property that for every infinite place  $v$  of  $F^+$ ,  $G(F_v^+)$  is compact.

It will be convenient to fix an integral model for  $G$ . We may find a nonzero  $N \in \mathcal{O}_{F^+}$  such that  $G$  admits a reductive (in particular, smooth) model  $\mathcal{G}$  over  $\mathcal{O}_{F^+}[1/N]$ , which furthermore admits an isomorphism  $\iota : \mathcal{G}/_{\mathcal{O}_F}[1/N] \xrightarrow{\sim} (\text{GL}_n)_{/\mathcal{O}_F}[1/N]$ . (We can obtain both by spreading out.)

If  $v$  is any finite place of  $F^+$  which splits in  $F$ , and if  $w$  is a place of  $F$  lying above  $v$ , then  $\iota_w$  gives rise to an isomorphism

$$\iota_w : G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_w),$$

which restricts to an isomorphism  $\mathcal{G}(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_w})$  if  $v$  does not divide  $N$ . If  $w^c$  is the other prime of  $F$  lying over  $v$ , then  $c$  induces an isomorphism  $c : \mathrm{GL}_n(F_w) \xrightarrow{\sim} \mathrm{GL}_n(F_{w^c})$ , and our assumption that  $G$  is an outer form of  $\mathrm{GL}_n$  ensures that  $c \circ \iota_w$  is conjugate (by an element of  $\mathrm{GL}_n(\mathcal{O}_{F_{w^c}})$ , if  $v \nmid N$ ) to the inverse transpose of  $\iota_{w^c}$ .

**7.1.2. Modular forms and Hecke operators away from  $p$ .** If  $W$  is an  $\overline{\mathbb{Z}}_p$ -module with an action of  $\mathcal{G}(\mathcal{O}_{F^+,p})$ , and if  $U$  is a compact open subgroup of  $G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p}) \subset G(\mathbb{A}_{F^+}^{\infty})$ , then we let  $S(U, W)$  denote the space of modular forms on  $G$  of level  $U$  with coefficients  $W$ , i.e. the space of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty}) \rightarrow W$$

such that  $f(gu) = u_p^{-1} f(g)$  for all  $u \in U$ . (Note that when  $W = W_\lambda$  with  $W_\lambda$  as in §7.1.4, these give examples of spaces of algebraic modular forms in the sense of [Gro99].)

For any  $U$  as above, we may write  $G(\mathbb{A}_{F^+}^{\infty}) = \coprod_i G(F^+) t_i U$  for some finite set  $\{t_i\}$ . Then there is an isomorphism

$$S(U, W) \rightarrow \bigoplus_i W^{U \cap t_i^{-1} G(F^+) t_i}$$

given by  $f \mapsto (f(t_i))_i$ . We say that  $U$  is *sufficiently small* if for some finite place  $v$  of  $F^+$  the projection of  $U$  to  $G(F_v^+)$  contains no element of finite order other than the identity. If  $U$  is sufficiently small, then for each  $i$  as above we have  $U \cap t_i^{-1} G(F^+) t_i = \{1\}$ , so for any  $W$  as above and any  $\overline{\mathbb{Z}}_p$ -algebra  $A$ , we have a natural isomorphism

$$(7.1.3) \quad S(U, W) \otimes_{\overline{\mathbb{Z}}_p} A \xrightarrow{\sim} S(U, W \otimes_{\overline{\mathbb{Z}}_p} A).$$

If  $U$  is any compact open subgroup as above, then we will say that  $U$  is *unramified* at a place  $v$  of  $F^+$  which splits in  $F$  and does not divide  $N$  if  $U = \mathcal{G}(\mathcal{O}_{F_v^+}) U^v$  for a compact open subgroup  $U^v$  of  $G(\mathbb{A}_{F^+}^{\infty,v})$ . (Recall that  $\mathcal{G}(\mathcal{O}_{F_v^+}) \cong \mathrm{GL}_n(\mathcal{O}_{F_w})$  via  $\iota_w$ , for either of the places  $w$  of  $F$  lying over  $v$ .) We will say that  $U$  is *unramified at  $p$*  if it is unramified at all places  $v|p$ . We note that  $U$  is unramified at all but finitely many places of  $F^+$  that split in  $F$ .

Given  $U$ , we let  $\mathcal{P}_U$  denote the set of finite places of  $F$  which lie above split places of  $F^+$  at which  $U$  is unramified and which do not divide  $pN$ , and we let  $\mathcal{P} \subset \mathcal{P}_U$  denote a subset with finite complement. We then define  $\mathbb{T}^{\mathcal{P}}$  as in Subsection 4.2. The algebra  $\mathbb{T}^{\mathcal{P}}$  acts on  $S(U, W)$  (for any  $\mathcal{G}(\mathcal{O}_{F^+,p})$ -module  $W$ ) via the Hecke operators

$$T_w^{(j)} := \iota_w^{-1} \left[ \mathrm{GL}_n(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_{F_w}) \right]$$

for  $w \in \mathcal{P}$  and  $\varpi_w$  a uniformiser in  $\mathcal{O}_{F_w}$ . Note that  $T_{w^c}^{(j)} = T_w^{(n-j)}(T_w^{(n)})^{-1}$  on  $S(U, W)$ .

7.1.4. *Modular forms of weight  $\lambda$ .* Let  $(\mathbb{Z}_+^n)^{\text{Hom}(F, \overline{\mathbb{Q}}_p)}$  be the elements of  $(\mathbb{Z}_+^n)^{\text{Hom}(F, \overline{\mathbb{Q}}_p)}$  with

$$\lambda_{\sigma, i} + \lambda_{\sigma c, n+1-i} = 0$$

for all  $\sigma \in \text{Hom}(F, \overline{\mathbb{Q}}_p)$  and  $1 \leq i \leq n$ . For any place  $w|p$  of  $F$  let  $\lambda_w$  denote the projection of  $\lambda$  to  $(\mathbb{Z}_+^n)^{\text{Hom}(F_w, \overline{\mathbb{Q}}_p)}$ . In §4.1.1 we defined a finite free  $\overline{\mathbb{Z}}_p$ -module  $W_{\lambda_w}$  with an action of  $\text{GL}_n(\mathcal{O}_{F_w})$ . We give this an action of  $\mathcal{G}(\mathcal{O}_{F_w^+})$  via  $\iota_w$ . This depends only on  $v := w|_{F^+}$ , as  $\iota_{w^c}$  is conjugate to the inverse-transpose of  $c \circ \iota_w$  (consider the formal characters of the  $M_{\lambda_\tau}$ ), so we will also denote it by  $W_{\lambda_v}$ . Let  $W_\lambda$  be the finite free  $\overline{\mathbb{Z}}_p$ -module with an action of  $\mathcal{G}(\mathcal{O}_{F^+, p})$  defined by

$$W_\lambda := \bigotimes_{v|p} W_{\lambda_v}.$$

If  $A$  is a  $\overline{\mathbb{Z}}_p$ -module and  $U$  is a compact open subgroup of  $G(\mathbb{A}_{F^+}^{\infty, p}) \times \mathcal{G}(\mathcal{O}_{F^+, p})$  that is unramified at  $p$ , then we let

$$S_\lambda(U, A) := S(U, W_\lambda \otimes_{\overline{\mathbb{Z}}_p} A).$$

From (7.1.3), we see that if  $U$  is sufficiently small, then there is a natural isomorphism

$$(7.1.5) \quad S_\lambda(U, \overline{\mathbb{Z}}_p) \otimes_{\overline{\mathbb{Z}}_p} A \xrightarrow{\sim} S_\lambda(U, A).$$

We now recall the relationship between our spaces of modular forms and the space of automorphic forms on  $G$ . Write  $S_\lambda(\overline{\mathbb{Q}}_p)$  for the direct limit of the spaces  $S_\lambda(U, \overline{\mathbb{Q}}_p)$  over compact open subgroups  $U$  of  $G(\mathbb{A}_{F^+}^{\infty, p}) \times \mathcal{G}(\mathcal{O}_{F^+, p})$  (with the transition maps being the inclusions  $S_\lambda(U, \overline{\mathbb{Q}}_p) \subset S_\lambda(V, \overline{\mathbb{Q}}_p)$  whenever  $V \subset U$ ). Concretely,  $S_\lambda(\overline{\mathbb{Q}}_p)$  is the set of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) \rightarrow W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$$

such that there is a compact open subgroup  $U$  of  $G(\mathbb{A}_{F^+}^{\infty, p}) \times \mathcal{G}(\mathcal{O}_{F^+, p})$  with

$$f(gu) = u_p^{-1} f(g)$$

for all  $u \in U$ ,  $g \in G(\mathbb{A}_{F^+}^\infty)$ . This space has a natural left action of  $G(\mathbb{A}_{F^+}^\infty)$  via

$$(g \cdot f)(h) := g_p f(hg).$$

Recall that we have fixed an isomorphism  $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ . Fix a set of embeddings  $\{\tilde{\tau} : F \rightarrow \overline{\mathbb{Q}}_p\}$  such that  $\text{Hom}(F, \overline{\mathbb{Q}}_p) = \{\tilde{\tau}\} \sqcup \{\tilde{\tau} \circ c\}$ . Let  $\sigma_\lambda$  denote the representation of  $G(F_\infty^+)$  given by  $\bigotimes_{\tilde{\tau}} M_{\lambda_{\tilde{\tau}}}(\mathbb{C})$ , where  $G(F_\infty^+)$  acts on  $M_{\lambda_{\tilde{\tau}}}(\mathbb{C})$  via the homomorphism  $G(F_\infty^+) \rightarrow G(\mathbb{C}) \xrightarrow{\iota} \text{GL}_n(\mathbb{C})$ , where both maps are induced by  $\iota \circ \tilde{\tau} : F \rightarrow \mathbb{C}$ . As usual,  $\sigma_\lambda$  does not depend on the choice of the  $\tilde{\tau}$  up to isomorphism. Let  $\mathcal{A}$  denote the space of automorphic

forms on  $G(F^+) \backslash G(\mathbb{A}_{F^+})$ . As in the proof of Proposition 3.3.2 of [CHT08] one easily obtains the following lemma.

**Lemma 7.1.6.** *There is an isomorphism of  $G(\mathbb{A}_{F^+}^\infty)$ -modules*

$$S_\lambda(\overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p, \iota} \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{G(F_\infty^+)}(\sigma_\lambda^\vee, \mathcal{A}).$$

In particular, we note that  $S_\lambda(\overline{\mathbb{Q}}_p)$  is a semisimple admissible  $G(\mathbb{A}_{F^+}^\infty)$ -module.

**7.2. Galois representations attached to modular forms.** In the following theorem we recall the results we will need regarding the existence of Galois representations attached to modular forms on the unitary group  $G$ . In the statement of the theorem, we let  $|\cdot|^{(1-n)/2} : F^\times \rightarrow \overline{\mathbb{Q}}_p^\times$  denote the unique square root of  $|\cdot|^{1-n}$  whose composite with  $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$  takes positive values. We will write  $\mathrm{rec}_w$  for  $\mathrm{rec}_{F_w}$ .

**Theorem 7.2.1.** *If  $\pi$  is an irreducible subrepresentation of the  $G(\mathbb{A}_{F^+}^\infty)$ -representation  $S_\lambda(\overline{\mathbb{Q}}_p)$ , then there is a continuous semisimple representation*

$$r_\pi : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

such that

- (i)  $r_\pi^c \cong r_\pi^\vee \otimes \varepsilon^{1-n}$ .
- (ii) *The representation  $r_\pi$  is de Rham, and is crystalline if  $\pi$  has level prime to  $p$ . If  $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$  then*

$$\mathrm{HT}_\tau(r_\pi) = \{\lambda_{\tau,1} + n - 1, \dots, \lambda_{\tau,n}\}.$$

- (iii) *If  $v \nmid p$  is a place of  $F^+$  which splits as  $v = ww^c$  in  $F$ , then*

$$\mathrm{WD}(r_\pi|_{G_{F_w}})^{\mathrm{F-ss}} \cong \mathrm{rec}_w((\pi_v \circ \iota_w^{-1}) \otimes |\cdot|^{(1-n)/2}).$$

- (iv) *If  $w|p$  is a place of  $F$ , write  $v = w|_{F^+}$ . Then*

$$\mathrm{WD}(r_\pi|_{G_{F_w}})^{\mathrm{ss}} \cong \mathrm{rec}_w((\pi_v \circ \iota_w^{-1}) \otimes |\cdot|^{(1-n)/2})^{\mathrm{ss}}.$$

*Proof.* In the case that  $F^+ \neq \mathbb{Q}$ , this follows from Lemma 7.1.6, Corollary 5.3 of [Lab09] and the main results of [Car10] and [BLGGT11]. As remarked in [Lab09], with some additional work Corollary 5.3 of [Lab09] may be extended to the case  $F^+ = \mathbb{Q}$ . In the case that  $G$  is quasisplit at all finite places, this extension is a special case of Theorem 6.1 of [Whi11]. However, the assumption that  $G$  is quasisplit at all finite places is not needed in the proof, cf. [Shi11]. Finally, we remark that in the case  $n = 3$  (which is the case that we will ultimately be interested in), it is presumably possible to extract these results from [Rog90].  $\square$

**7.3. Serre weights.** We now wish to define what it means for an irreducible representation  $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$  to be modular of some weight. The basic notion required for this is that of a *Serre weight*, which we now define.

**Definition 7.3.1.** A *Serre weight* is an isomorphism class of irreducible smooth  $\overline{\mathbb{F}}_p$ -representations of  $\mathcal{G}(\mathcal{O}_{F^+, p})$ .

*Remark 7.3.2.* We will engage in the same abuse of language with regard to Serre weights as the one that was signalled in the case of (general) weights in Remark 2.1.2.

The kernel of the natural map  $\mathcal{G}(\mathcal{O}_{F^+, p}) \rightarrow \prod_{v|p} \mathcal{G}(k_v)$ , where  $k_v$  denotes the residue field of  $F_v^+$ , is a normal pro- $p$  group, and so a Serre weight may equally well be regarded as an isomorphism class of irreducible representations of the product  $\prod_{v|p} \mathcal{G}(k_v)$ . If we fix a prime  $\tilde{v}$  of  $F$  lying over  $v$  for each  $v$  lying over  $p$ , then the isomorphisms  $\iota_{\tilde{v}}$  induce an isomorphism

$$(7.3.3) \quad \prod_{v|p} \mathcal{G}(k_v) \xrightarrow{\sim} \prod_{v|p} \mathrm{GL}_n(k_{\tilde{v}}),$$

and hence an irreducible representation of this product is nothing but a tensor product of irreducible representations of the groups  $\mathrm{GL}_n(k_{\tilde{v}})$ . This relates the concept of Serre weight to the general notion of weight discussed in Subsections 2.1 and in §4.1.2. This allows us to give an “explicit” description of all Serre weights, as we now explain.

First note that if  $w|p$ , then  $c$  induces an isomorphism  $k_w \xrightarrow{\sim} k_{w^c}$ , and hence a bijection  $\mathrm{Hom}(k_{w^c}, \overline{\mathbb{F}}_p) \xrightarrow{\sim} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  via  $\sigma \mapsto \sigma c$ . In this way we obtain an action of  $c$  on  $(\mathbb{Z}_+^n) \prod_{w|p} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$ . We let  $(\mathbb{Z}_+^n)_0 \prod_{w|p} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  denote the subset of  $(\mathbb{Z}_+^n) \prod_{w|p} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  consisting of elements  $a$  such that for each  $w|p$ , if  $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  and  $1 \leq i \leq n$  then

$$a_{\sigma, i} + a_{\sigma c, n+1-i} = 0.$$

We say that an element  $a \in (\mathbb{Z}_+^n)_0 \prod_{w|p} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  is a *restricted weight* if, for each  $w|p$ , the projection  $a_w \in (\mathbb{Z}_+^n) \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  is restricted in the sense of §4.1.2; that is, if, for each  $w|p$  and each  $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$ , we have

$$a_{\sigma, i} - a_{\sigma, i+1} \leq p - 1$$

for all  $1 \leq i \leq n - 1$ .

Let  $a \in (\mathbb{Z}_+^n)_0 \prod_{w|p} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$  be a restricted weight. For any place  $w|p$  of  $F$  we defined in §4.1.2 a finite free  $\overline{\mathbb{F}}_p$ -module  $F_{a_w}$  with an action of  $\mathrm{GL}_n(k_w)$ . We give this an action of  $\mathcal{G}(k_v)$  via  $\iota_w$ . Just as in the case of  $W_\lambda$  in §7.1.4, this depends only on  $v := w|_{F^+}$ , so we will also denote it by  $F_{a_v}$ .

We then define the Serre weight  $F_a$  to be the tensor product

$$F_a := \bigotimes_{v|p} F_{a_v},$$

thought of as an irreducible representation of  $\prod_{v|p} \mathcal{G}(k_v)$ , or equivalently, of  $\mathcal{G}(\mathcal{O}_{F^+,p})$ .

**Lemma 7.3.4.** *Every Serre weight admits a representative of the form  $F_a$ , for some restricted weight  $a \in (\mathbb{Z}_+^n)_{\prod_{w|p} \text{Hom}(k_w, \overline{\mathbb{F}}_p)}$ .*

*Proof.* This follows from the isomorphism (7.3.3), together with the fact, noted in §4.1.2, that every irreducible representation of  $\text{GL}_n(k_w)$  is of the form  $F_{a_w}$  for some restricted weight  $a_w \in (\mathbb{Z}_+^n)^{\text{Hom}(k_w, \overline{\mathbb{F}}_p)}$ .  $\square$

By virtue of this lemma (and taking into account Remark 7.3.2), from now on we will typically denote a Serre weight by  $F_a$ , for some restricted weight  $a \in (\mathbb{Z}_+^n)_{\prod_{w|p} \text{Hom}(k_w, \overline{\mathbb{F}}_p)}$ .

If  $p$  is unramified in  $F$ , then we may introduce the notion of the *lift* of a restricted weight. Namely, if  $p$  is unramified in  $F$ , so that  $\text{Hom}(F, \overline{\mathbb{Q}}_p) = \prod_{w|p} \text{Hom}(F_w, \overline{\mathbb{Q}}_p)$  and  $\text{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p) = \prod_{w|p} \text{Hom}(k_w, \overline{\mathbb{F}}_p)$  are in natural bijection, then we say that a weight  $\lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(F, \overline{\mathbb{Q}}_p)}$  is the *lift* of a restricted weight  $a$  if for each  $w|p$ , we have that  $\lambda_w$  is the lift of  $a_w$  in the sense of §4.1.2; i.e. if for each  $w|p$  and each  $\tau \in \text{Hom}(F_w, \overline{\mathbb{Q}}_p)$ , we have  $\lambda_\tau = a_\tau$ .

Suppose now that  $\bar{r} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$  is a continuous irreducible representation. If  $U$  is any compact open subgroup of  $G(\mathbb{A}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$  for which  $\bar{r}$  is unramified at all the places lying in  $\mathcal{P}$ , then we may associate a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}^{\mathcal{P}}$  to  $\bar{r}$ , as in Subsection 4.2. (We remark that, conversely,  $\mathfrak{m}$  determines  $\bar{r}$  by the Chebotarev density theorem.)

*Definition 7.3.5.* If  $\bar{r} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$  is continuous and irreducible, then we say that  $\bar{r}$  is *modular of Serre weight  $F_a$* , if there is a compact open subset  $U$  of  $G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$  which is unramified at  $p$ , a subset  $\mathcal{P} \subset \mathcal{P}_U$  with finite complement such that  $\bar{r}$  is unramified at all the places lying in  $\mathcal{P}$ , and such that, if  $\mathfrak{m}$  is the maximal ideal in  $\mathbb{T}^{\mathcal{P}}$  associated to  $\bar{r}$ , then  $S(U, F_a)_{\mathfrak{m}} \neq 0$ .

We say that  $\bar{r}$  is *modular* if it is modular of some Serre weight.

We write  $W(\bar{r}) := \{F_a \mid \bar{r} \text{ is modular of Serre weight } F_a\}$ , and refer to this as the set of Serre weights of  $\bar{r}$ . (Thus  $\bar{r}$  is modular if and only if  $W(\bar{r}) \neq \emptyset$ .)

*Remark 7.3.6.* If, in the context of the preceding definition, we have that  $S(U, F_a)_{\mathfrak{m}} \neq 0$ , then certainly  $S(U', F_a)_{\mathfrak{m}} \neq 0$  for any compact open subgroup  $U'$  of  $U$ . (Here we have assumed, without loss of generality, that  $\mathcal{P} \subset \mathcal{P}_{U'} \subset \mathcal{P}_U$ , so that  $\mathbb{T}^{\mathcal{P}}$  acts on both spaces of modular forms.) Thus, when working in the context of this definition, we may assume that the level  $U$  that appears is sufficiently small, and we will assume this from now on.

*Remark 7.3.7.* Note that if  $\bar{r} : G_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$  is modular then  $\bar{r}^c \cong \bar{r}^\vee \bar{\varepsilon}^{n-1}$ .

**7.4. Verifying the characteristic zero axioms.** We now apply the theory developed in the earlier sections of this paper to the specific situation of the unitary groups considered earlier in this section. To this end, we will define appropriate spaces  $S$  and  $\tilde{S}$  which satisfy the axioms of Section 4. We maintain the notation established earlier in this section, and furthermore assume that  $p$  is unramified in  $F$ .

Assume that  $\bar{r}$  is modular of Serre weight  $F_a$ , in the sense of Definition 7.3.5, for some restricted weight  $a \in (\mathbb{Z}_+^n)_{\prod_{w|p} \text{Hom}(k_w, \bar{\mathbb{F}}_p)}$ . Then, by definition, there is a compact open subgroup  $U$  of  $G(\mathbb{A}_{F^+}^{\infty,p}) \times \mathcal{G}(\mathcal{O}_{F^+,p})$  that is unramified at  $p$  and a subset  $\mathcal{P} \subset \mathcal{P}_U$  with finite complement such that  $S(U, F_a)_{\mathfrak{m}} \neq 0$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathbb{T}^{\mathcal{P}}$  corresponding to  $\bar{r}$ . Shrinking  $\mathcal{P}$  if necessary, by Remark 7.3.6 we may (and do) assume that  $U$  is sufficiently small.

We fix a place  $w|p$  of  $F$ , and assume that  $\bar{r}|_{G_{F_w}}$  is irreducible. We write  $v = w|_{F^+}$ , and write  $U = U_v \times U^v$ , where  $U^v \subset G(\mathbb{A}_{F^+}^{\infty,v})$  is compact open.

We now set up some further notation in preparation for defining the spaces  $S$  and  $\tilde{S}$ .

If  $W$  is any  $\bar{\mathbb{Z}}_p$ -module with an action of  $\mathcal{G}(\mathcal{O}_{F^+,p})$  such that  $\mathcal{G}(\mathcal{O}_{F_v^+})$  acts trivially, we define  $S(U^v, W)$  to be the space of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty}) \rightarrow W$$

such that  $f(gu) = u_p^{-1}f(g)$  for all  $g \in G(\mathbb{A}_{F^+}^{\infty})$  and all  $u \in U^v$ . It is an  $\bar{\mathbb{Z}}_p$ -module with an action of  $G(F_v^+)$ , given by  $(\gamma \cdot f)(g) = f(g\gamma)$  for  $\gamma \in G(F_v^+)$  and  $g \in G(\mathbb{A}_{F^+}^{\infty})$ , and a commuting action of  $\mathbb{T}^{\mathcal{P}}$ . Let  $S^{\text{sm}}(U^v, W)$  be the largest smooth  $G(F_v^+)$ -subrepresentation of  $S(U^v, W)$ , i.e.

$$S^{\text{sm}}(U^v, W) = \varinjlim_{U'_v \leq U^v \text{ open}} S(U^v, W)^{U'_v}.$$

If  $W$  is a finite free  $\bar{\mathbb{Z}}_p$ -module with an action of  $\mathcal{G}(\mathcal{O}_{F^+,p})$  such that  $\mathcal{G}(\mathcal{O}_{F_v^+})$  acts trivially, we define  $S^{\text{lg}}(U^v, W)$  to be the space of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+}^{\infty}) \rightarrow W$$

such that  $f(gu) = u_p^{-1}f(g)$  for all  $g \in G(\mathbb{A}_{F^+}^{\infty})$  and all  $u \in U^v$  and such that for all  $g \in G(\mathbb{A}_{F^+}^{\infty})$  and all  $\eta \in W^{\vee}$  the map  $G(F_v^+) \rightarrow \bar{\mathbb{Z}}_p$ ,  $\gamma \mapsto \langle f(g\gamma), \eta \rangle$  is locally algebraic. Note that  $S^{\text{lg}}(U^v, W)$  has natural, commuting actions of  $G(F_v^+)$  and  $\mathbb{T}^{\mathcal{P}}$ .

We let  $S = S^{\text{sm}}(U^v, \otimes_{v' \neq v} F_{a_{v'}})$  and  $\tilde{S} = S^{\text{lg}}(U^v, \otimes_{v' \neq v} W_{\lambda_{v'}})$ , where  $\lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(F, \bar{\mathbb{Q}}_p)}$  denotes the lift of  $a$ . Note that these spaces do not depend on  $a_w$  and  $\lambda_w$ . We identify  $G(F_v^+)$  with  $\text{GL}_n(F_w)$  via  $\iota_w$ . Then  $S$  and  $\tilde{S}$  have commuting actions of  $\mathbb{T}$  and  $\text{GL}_n(F_w)$ , and the action of  $\text{GL}_n(F_w)$  on  $S$  is smooth.

**Lemma 7.4.1.** *We have a natural isomorphism*

$$\tilde{S} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p \xrightarrow{\sim} S^{\text{sm}}(U^v, \otimes_{v' \neq v} (W_{\lambda_{v'}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p))$$

that is compatible with the  $\mathbb{T}$ - and  $\text{GL}_n(F_w)$ -actions.

*Proof.* Note first that  $G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty)$  is compact, as  $G$  is anisotropic at infinity. The above map is surjective, since any element of the right-hand side takes only finitely many values so that we can lift it to a smooth element of  $\tilde{S}$ .

Let  $f \in \tilde{S}$  be such that  $f \otimes 1$  is in the kernel of the map. Let  $e_1^\vee, \dots, e_d^\vee$  denote a basis of  $(\otimes_{v' \neq v} W_{\lambda_{v'}})^\vee$ . By compactness of  $G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty)$  and by definition of  $\tilde{S}$  we can find a compact open subgroup  $U'_v \subset \text{GL}_n(\mathcal{O}_{F_w})$  with the following property: writing  $G(\mathbb{A}_{F^+}^\infty) = \prod_{i=1}^r G(F^+) g_i U'_v U^v$  there are algebraic functions  $f_{ij} : U'_v \rightarrow \mathbb{Z}_p$  such that  $\langle f(g_i u'_v), e_j^\vee \rangle = f_{ij}(u'_v)$  for all  $i$  and all  $u'_v \in U'_v$ . Since  $\text{Res}_{F_w/\mathbb{Q}_p} \text{GL}_n$  is defined over  $\mathbb{Q}_p$ , there is a finite extension  $K$  of  $\mathbb{Q}_p$  with uniformiser  $\varpi_K$  such that all  $f_{ij}$  take values in  $\mathcal{O}_K$ . Since  $f \otimes 1$  is in the kernel, the  $f_{ij}$  take values in  $\varpi_K \mathcal{O}_K$ . It follows that  $\varpi_K^{-1} f \in \tilde{S}$ , so  $f \otimes 1 = 0$ .  $\square$

By the lemma we get an embedding  $S \hookrightarrow \tilde{S} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$  that is compatible with the  $\mathbb{T}$ - and  $\text{GL}_n(F_w)$ -actions.

**Lemma 7.4.2.** *If  $\tilde{V}$  is a free finite-rank  $\overline{\mathbb{Z}}_p$ -module equipped with a locally algebraic action of  $\text{GL}_n(\mathcal{O}_{F_w})$  and  $M$  is a finite free  $\overline{\mathbb{Z}}_p$ -module with an action of  $\prod_{v' \neq v} \mathcal{G}(\mathcal{O}_{F_v^+})$ , then there is a natural isomorphism of  $\mathbb{T}$ -modules*

$$(\tilde{V} \otimes_{\overline{\mathbb{Z}}_p} S^{\text{alg}}(U^v, M))^{\text{GL}_n(\mathcal{O}_{F_w})} \xrightarrow{\sim} S(U, \tilde{V} \otimes_{\overline{\mathbb{Z}}_p} M).$$

*Proof.* Note that we have a  $\mathbb{T}$ - and  $\text{GL}_n(F_w)$ -equivariant map  $\eta : \tilde{V} \otimes S(U^v, M) \rightarrow S(U^v, \tilde{V} \otimes M)$  given by  $x \otimes f_0 \mapsto f(g) := x \otimes f_0(g)$ . It is an isomorphism since  $\tilde{V}$  is finite free as  $\overline{\mathbb{Z}}_p$ -module. By restricting to locally algebraic functions on the left-hand side and by passing to  $\text{GL}_n(\mathcal{O}_{F_w})$ -invariants, we get a map  $\eta'$  as in the statement of the lemma that is moreover injective. Pick a basis  $e_1, \dots, e_d$  of  $\tilde{V}$ . Given any element  $f \in S(U, \tilde{V} \otimes_{\overline{\mathbb{Z}}_p} M)$ , write  $\eta^{-1}(f) = \sum e_i \otimes f_i$  for unique  $f_i \in S(U^v, M)$ . To show that  $\eta'$  is surjective, we need to prove that  $f_i \in S^{\text{alg}}(U^v, M)$  for all  $i$ . But a simple computation, using the fact that  $f$  is  $\text{GL}_n(\mathcal{O}_{F_w})$ -invariant, shows that for fixed  $g \in G(\mathbb{A}_{F^+}^\infty)$  and  $m^\vee \in M^\vee$ , the function  $\gamma \mapsto \langle f_i(g\gamma), m^\vee \rangle$  on  $\text{GL}_n(\mathcal{O}_{F_w})$  is a linear combination of matrix coefficients of  $\tilde{V}$ , hence locally algebraic.  $\square$

A similar but easier argument establishes the following lemma.

**Lemma 7.4.3.** *If  $V$  is a free finite-rank  $\overline{\mathbb{F}}_p$ -module equipped with a smooth action of  $\text{GL}_n(\mathcal{O}_{F_w})$  and  $M$  is an  $\overline{\mathbb{F}}_p$ -module with an action of  $\prod_{v' \neq v} \mathcal{G}(\mathcal{O}_{F_v^+})$ , then there is a natural isomorphism of  $\mathbb{T}$ -modules*

$$(V \otimes_{\overline{\mathbb{F}}_p} S^{\text{sm}}(U^v, M))^{\text{GL}_n(\mathcal{O}_{F_w})} \xrightarrow{\sim} S(U, V \otimes_{\overline{\mathbb{F}}_p} M).$$

**Proposition 7.4.4.** *In the present setting, Axioms  $\tilde{\text{A}}1$ – $\tilde{\text{A}}3$  hold.*

*Proof.* To verify Axiom  $\tilde{\text{A}}1$ , first note that  $\tilde{S}(\tilde{V})$  is finite free by Lemma 7.4.2 (as  $U$  is sufficiently small). It remains to show that

$$(\tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \tilde{S})^{\text{GL}_n(\mathcal{O}_{F_w})} \otimes_{\tilde{\mathbb{Z}}_p} A \xrightarrow{\sim} (\tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \tilde{S} \otimes_{\tilde{\mathbb{Z}}_p} A)^{\text{GL}_n(\mathcal{O}_{F_w})}$$

when  $A = \overline{\mathbb{F}}_p$  or  $A = \overline{\mathbb{Q}}_p$ . In the first case we can rewrite the left-hand side using Lemmas 7.4.2, 7.4.3, and (7.1.3) as follows:

$$\begin{aligned} & (\tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \tilde{S})^{\text{GL}_n(\mathcal{O}_{F_w})} \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \\ & \cong S(U, \tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \bigotimes_{v' \neq v} W_{\lambda_{v'}}) \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{F}}_p \\ & \cong S(U, \tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \bigotimes_{v' \neq v} W_{\lambda_{v'}} \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{F}}_p) \\ & \cong ((\tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{F}}_p) \otimes_{\overline{\mathbb{F}}_p} S^{\text{sm}}(U^v, \otimes_{v' \neq v} (W_{\lambda_{v'}} \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{F}}_p)))^{\text{GL}_n(\mathcal{O}_{F_w})} \\ & \cong (\tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \tilde{S} \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{F}}_p)^{\text{GL}_n(\mathcal{O}_{F_w})}, \end{aligned}$$

where the last isomorphism follows from Lemma 7.4.1. In the second case note that the map is injective because  $\overline{\mathbb{Q}}_p$  is  $\tilde{\mathbb{Z}}_p$ -flat. Any element of the right-hand side can be written as  $m \otimes \lambda$ , where  $m \in \tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \tilde{S}$  and  $\lambda \in \overline{\mathbb{Q}}_p^\times$ . Since  $\tilde{V} \otimes_{\tilde{\mathbb{Z}}_p} \tilde{S}$  is  $\tilde{\mathbb{Z}}_p$ -torsion free it follows that  $m$  is  $\text{GL}_n(\mathcal{O}_{F_w})$ -stable, as required.

To verify Axiom  $\tilde{\text{A}}2$ , let  $\lambda_w \in (\mathbb{Z}_+^n)^{\text{Hom}(F_w, \overline{\mathbb{Q}}_p)}$ . Our definition of  $\tilde{S}$  only depended on  $\lambda$  away from  $w$ ,  $w^c$ , so we are free to modify  $\lambda$  at  $w$  and  $w^c$  so that it agrees with the given  $\lambda_w$  at  $w$ . By Lemma 7.4.2 and Axiom  $\tilde{\text{A}}1$  we have  $\tilde{S}(W_{\lambda_w} \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p) = S(U, W_\lambda \otimes_{\tilde{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p)$ , so by assumption  $S_\lambda(U, \overline{\mathbb{Q}}_p)_m \neq 0$ . It follows from Lemma 4.5.1, as well as Theorem 7.2.1(iii) together with Corollary 3.1.2 of [CHT08] and the Chebotarev density theorem that there is an irreducible  $G(\mathbb{A}_{F_+}^\infty)$ -subrepresentation  $\pi$  of  $S_\lambda(\overline{\mathbb{Q}}_p)$  with  $\bar{r}_\pi \cong \bar{r}^\vee$ . By Theorem 7.2.1,  $r_\pi|_{G_{F_w}}$  is crystalline with  $\text{HT}_\tau(r_\pi|_{G_{F_w}}) = (\lambda_{\tau,1} + n - 1, \dots, \lambda_{\tau,n})$  for all  $\tau : F_w \rightarrow \overline{\mathbb{Q}}_p$ , so  $r_\pi^\vee|_{G_{F_w}}$  provides the required lift. The claim about the characteristic polynomial of  $\varphi^{[F_w:\mathbb{Q}_p]}$  follows from Theorem 7.2.1(iv) together with Corollary 3.1.2 of [CHT08].

It remains to verify Axiom  $\tilde{\text{A}}3$ . As our definition of  $\tilde{S}$  only depended on  $\lambda$  away from  $w$ ,  $w^c$ , we are free to modify  $\lambda$  at  $w$  and  $w^c$  so that  $\lambda_w = 0$  at  $w$ . By choosing a  $\text{GL}_n(k_w)$ -stable lattice  $R_0$  in  $R$  and applying Axiom  $\tilde{\text{A}}1$  and Lemma 7.4.2 to the smooth representation  $\tilde{V} = R_0$  we have  $\tilde{S}(R) \cong$

$S(U, R \otimes_{\overline{\mathbb{Q}}_p} (W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p))$ . Therefore

$$\begin{aligned} \tilde{S}(R)_m &\cong S(U, R \otimes_{\overline{\mathbb{Q}}_p} (W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p))_m \\ &\cong S(U', R \otimes_{\overline{\mathbb{Q}}_p} (W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p))_m^{U/U'} \\ &\cong \text{Hom}_{U_v} (R^\vee, S(U', W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p)_m), \end{aligned}$$

where  $U'$  is the kernel of the map  $U \rightarrow U_v \xrightarrow{\iota_w} \text{GL}_n(\mathcal{O}_{F_w}) \rightarrow \text{GL}_n(k_w)$ . As is in the preceding paragraph there is thus an irreducible subrepresentation  $\pi$  of  $S_\lambda(\overline{\mathbb{Q}}_p)$  such that  $\pi^{U'} \cap S(U', W_\lambda \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p)_m \neq 0$  and  $\pi|_{\mathcal{G}(\mathcal{O}_{F_v^+})}$  contains  $R^\vee$ . In particular,  $\bar{r}_\pi \cong \bar{r}^\vee$ . Since  $\lambda_w = 0$ , it follows from Theorem 7.2.1 and Proposition 2.4.1 that  $r_\pi^\vee|_{G_{F_w}}$  provides the required lift.  $\square$

**7.5. The main theorem for definite unitary groups.** From now on we assume that  $n = 3$  and that  $p$  splits in  $F$ .

The set  $W^?(r)$  of predicted Serre weights for  $r$  is defined in terms of the set of predicted weights  $W^?(r|_{G_{F_w}})$  for the representations  $r|_{G_{F_w}}$  for  $w|p$  a place of  $F$  (see Subsection 5.1). Namely, if  $v$  is a place of  $F^+$  lying over  $p$ , if  $w$  is a place of  $F$  lying over  $v$ , if  $V_v$  is an irreducible representation of  $\mathcal{G}(k_v)$  over  $\overline{\mathbb{F}}_p$ , and if  $w$  is a place of  $F$  lying over  $v$ , then we introduce the following condition:

$$(7.5.1) \quad V_v \circ \iota_w^{-1} \in W_w^?(r|_{G_{F_w}}).$$

**Definition 7.5.2.** We define the set of predicted Serre weights for  $r$  to be

$$W^?(r) := \{V = \otimes_{v|p} V_v : V_v \text{ satisfies condition (7.5.1) for all } w|v|p\}.$$

*Remark 7.5.3.* In Definition 7.5.2, it suffices to fix a place  $\tilde{v}$  over each place  $v|p$  of  $F^+$ , and then to impose the condition (7.5.1) just at these chosen places  $\tilde{v}$ . This follows from the facts that  $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}^2$ , that  $W_w^?(r|_{G_{F_w}})^\vee = W_w^?(r|_{G_{F_w}}^\vee \otimes \bar{\epsilon}^2)$  (by Proposition 6.23 in [Her09]) and that  $\iota_w^c$  is conjugate to the inverse transpose of  $c \circ \iota_w$ . Indeed, taken together, these imply that imposing condition (7.5.1) at  $\tilde{v}$  is equivalent to imposing it at  $\tilde{v}^c$ .

*Definition 7.5.4.* We say that a Serre weight  $V = \otimes_{v|p} V_v$  is *generic* (resp. *strongly generic*) if  $V_v \circ \iota_w^{-1}$  is a generic (resp. strongly generic) weight in the sense of Definition 6.2.2 for all places  $w|p$  of  $F$ . We let  $W_{\text{gen}}(r)$  denote the set of generic Serre weights in  $W(r)$ .

**Theorem 7.5.5.** *Let  $F$  be an imaginary CM field in which the prime  $p$  splits completely, and suppose that  $\bar{r} : G_F \rightarrow \text{GL}_3(\overline{\mathbb{F}}_p)$  is a continuous representation. For all places  $w|p$  of  $F$  suppose that  $\bar{r}|_{G_{F_w}}$  is irreducible. Then  $W_{\text{gen}}(r) \subset W^?(r)$ .*

*Proof.* Fix a Serre weight  $V = \otimes_{v|p} V_v \in W_{\text{gen}}(r)$ . Since the definition of  $W^?(r)$  is local, we need only prove that if  $w|p$  is a place of  $F$  with  $w|_{F^+} = v$ , then  $V_v \circ \iota_w^{-1} \in W_w^?(r|_{G_{F_w}})$ . We set up the axiomatic framework as in

Subsection 7.4 using the place  $w$  and the Serre weight  $V$ . Then we see by Lemma 7.4.3 that  $V_v \circ \iota_w^{-1} \in W_w(\bar{r})$ . We can now conclude by Proposition 7.4.4 and Theorems 4.3.3, 5.2.5.  $\square$

**Theorem 7.5.6.** *Let  $F$  be an imaginary CM field in which the prime  $p$  splits completely, and suppose that  $\bar{r} : G_F \rightarrow \mathrm{GL}_3(\bar{\mathbb{F}}_p)$  is a continuous representation. For all places  $w|p$  of  $F$  suppose that  $\bar{r}|_{G_{F_w}}$  is irreducible. If  $\bar{r}$  is modular of some strongly generic Serre weight, then  $W_{\mathrm{gen}}(\bar{r}) = W^2(\bar{r})$ .*

*Proof.* By Theorem 7.5.5 we have  $W_{\mathrm{gen}}(\bar{r}) \subset W^2(\bar{r})$ . By assumption there is some  $V = \otimes_{v|p} V_v \in W_{\mathrm{gen}}(\bar{r})$  such that  $\bar{r}$  is modular of Serre weight  $V$ . Fix some place  $w|p$  of  $F$ . Arguing with each such  $w$  in turn, one over each place  $v|p$  of  $F^+$ , and iterating, we see that it suffices to show that if  $V'_v \circ \iota_w^{-1} \in W_w^2(\bar{r}|_{G_{F_w}})$  then  $(\otimes_{v' \neq v} V_{v'}) \otimes V'_v \in W(\bar{r})$ . Setting up the axiomatic framework as in Subsection 7.4 using the place  $w$  and the Serre weight  $V$ , we see that this follows from Proposition 7.4.4 and Theorems 4.3.3, 6.2.3.  $\square$

**7.6. Automorphic induction.** In this brief section we show that one can construct many examples in which our main theorem (Theorem 6.2.3) applies. Firstly, choose a CM field  $F$  with maximal totally real subfield  $F^+$  such that  $F/F^+$  is unramified at all finite places and such that  $p$  splits completely in  $F$ . (For example, choose any CM field  $F$  in which  $p$  splits completely, and use Lemma 4.1.2 of [CHT08] to find a Galois totally real field  $K^+$  in which  $p$  splits completely such that the extension  $FK^+/F^+K^+$  is everywhere unramified. The condition that  $F/F^+$  is unramified at all finite places is used below in order to apply Theorem 5.4 of [Lab09].) Fix a weight  $\lambda \in (\mathbb{Z}_+^3)_0^{\mathrm{Hom}(F, \bar{\mathbb{Q}}_p)}$ .

By Theorem 6 of chapter 10 of [AT09] (and its proof, which shows that we may assume that  $p$  is unramified in the extension), we may choose a cyclic totally real extension  $L^+/F^+$  of degree 3 in which every place of  $F^+$  lying over  $p$  is inert. Let  $L = FL^+$ . By Lemma 4.1.5 of [CHT08], we may choose a crystalline character  $\psi : G_L \rightarrow \bar{\mathbb{Q}}_p^\times$  such that

- $\psi\psi^c = \varepsilon^{-2}$ ,
- if  $w|p$  is a place of  $F$  and  $\tau : F_w \hookrightarrow \bar{\mathbb{Q}}_p$ , then

$$\bigcup \mathrm{HT}_{\bar{\tau}}(\psi|_{G_{L_w}}) = \{\lambda_{\tau,1} + 2, \lambda_{\tau,2} + 1, \lambda_{\tau,3}\},$$

where  $\bar{\tau}$  runs through all embeddings  $L_w \hookrightarrow \bar{\mathbb{Q}}_p$  such that  $\bar{\tau}|_{F_w} = \tau$ .

Assume furthermore that for each  $\tau$  we have  $\lambda_{\tau,1} - \lambda_{\tau,3} \leq p - 2$ . Set  $r_\psi := \mathrm{Ind}_{G_L}^{G_F} \psi$ . Then for each place  $w|p$  of  $F$  the representation  $\bar{r}_\psi^\vee|_{G_{F_w}}$  is irreducible (use the bounds on the  $\lambda_{\tau,i}$  and the assumption that each place of  $F^+$  lying over  $p$  is inert in  $L^+$ ), so in particular the global representation  $\bar{r}_\psi^\vee$  is irreducible. Then by Theorems 4.2 and 5.1 of [AC89], together with Theorem 5.4 of [Lab09] and Lemma 7.1.6 above, it follows that  $\bar{r}_\psi^\vee$  is modular of Serre weight  $F_a$ , where  $a \in (\mathbb{Z}_+^3)_{\prod_{w|p} \mathrm{Hom}(k_w, \bar{\mathbb{F}}_p)}$  is the

unique restricted weight of which  $\lambda$  is the lift. (Note that  $W_a \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p \cong F_a$  due to our bounds on the  $\lambda_{\tau,i}$ , for example by Proposition 3.18 in [Her09]. Note also that there is a unitary group satisfying our assumptions which is furthermore quasisplit at all finite places, cf. Section 2.1 of [Ger09].)

Thus if we choose any restricted weight  $a \in (\mathbb{Z}_+^3)_0^{\prod_{w|p} \text{Hom}(k_w, \overline{\mathbb{F}}_p)}$  such that  $F_{a_w}$  is strongly generic and in the lower alcove for each  $w|p$ , the representation  $\tilde{r}_\psi^\vee$  constructed above (with  $\lambda$  the lift of  $a$ ) satisfies the assumptions of Theorem 6.2.3. (Note that the condition that  $\lambda_{\tau,1} - \lambda_{\tau,3} \leq p-2$  is automatic from the assumption that all  $F_{a_w}$  are strongly generic.)

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