POTENTIAL AUTOMORPHY AND CHANGE OF WEIGHT.

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Abstract. We prove a new automorphy lifting theorem for l-adic representations where we impose a new condition at l, which we call 'potential diagonalizability'. This result allows for 'change of weight' and seems to be substantially more flexible than previous theorems along the same lines. We derive several applications. For instance we show that any irreducible, odd, essentially self-dual, regular, weakly compatible system of l-adic representations of the absolute Galois group of a totally real field is potentially automorphic, and hence is pure and its L-function has meromorphic continuation to the whole complex plane and satisfies the expected functional equation.

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Introduction.

Suppose that F and M are number fields, that S is a finite set of primes of F and that n is a positive integer. By a *weakly compatible system* of n-dimensional l-adic representations of G_F defined over M and unramified outside S we shall mean a family of continuous semi-simple representations

$$r_\lambda : G_F \longrightarrow GL_n(\overline{M}_\lambda),$$

where λ runs over the finite places of M, with the following properties.

- If $v \notin S$ is a finite place of F, then for all λ not dividing the residue characteristic of v, the representation r_λ is unramified at v and the characteristic polynomial of $r_\lambda(\mathrm{Frob}_v)$ lies in $M[X]$ and is independent of λ.
- Each representation r_λ is de Rham at all places above the residue characteristic of λ, and in fact crystalline at any place $v \notin S$ which divides the residue characteristic of λ.
- For each embedding $\tau : F \hookrightarrow \overline{M}$ the τ-Hodge-Tate numbers of r_λ are independent of λ.

In this paper we prove the following theorem (see Theorem 5.3.1).

Theorem A. *Let $\{r_\lambda\}$ be a weakly compatible system of n-dimensional l-adic representations of G_F defined over M and unramified outside S, where for simplicity we assume that M contains the image of each embedding $F \hookrightarrow \overline{M}$. Suppose that $\{r_\lambda\}$ satisfies the following properties.*

1. **(Irreducibility)** *Each r_λ is irreducible.*
2. **(Regularity)** *For each embedding $\tau : F \hookrightarrow M$ the representation r_λ has n distinct τ-Hodge-Tate numbers.*
3. **(Odd essential self-duality)** *F is totally real; and either each r_λ factors through a map to $GSp_n(\overline{M}_\lambda)$ with a totally odd multiplier character; or each r_λ factors through a map to $GO_n(\overline{M}_\lambda)$ with a totally even multiplier character. Moreover in either case the multiplier characters form a weakly compatible system.*

Then there is a finite, Galois, totally real extension over which all the r_λ's become automorphic. In particular for any embedding $\imath : M \hookrightarrow \mathbb{C}$ the partial L-function $L^S(\imath\{r_\lambda\}, s)$ converges in some right half plane and has meromorphic continuation to the whole complex plane.

This is not the first paper to prove potential automorphy results for compatible systems of l-adic representations of dimension greater than 2, see for example [HSBT10], [BLGHT09], [BLGG09]. However previous attempts only applied to very specific, though well known, examples (e.g. symmetric powers of the Tate modules of elliptic curves) and one had to exploit special properties of these examples. We believe this is the first general potential automorphy theorem in dimension greater than 2, and we are hopeful that it can be applied to many examples. We give an analogous theorem when F is an imaginary CM field. Other than this we do not see how to improve much on this theorem using current methods.

As one example application, suppose that $\mathcal{K}$ is a finite set of positive integers such that the $2^{\#\mathcal{K}}$ possible partial sums of elements of $\mathcal{K}$ are all distinct. For each $k \in \mathcal{K}$ let f_k be an elliptic modular newform of weight $k+1$ without complex multiplication. Then the $\#\mathcal{K}$-fold tensor product of the l-adic representations associated to the

f_k is potentially automorphic and the $\#\mathcal{K}$-fold product L-function for the f_k has meromorphic continuation to the whole complex plane. (See Corollary 5.3.3.)

The proof of Theorem A follows familiar lines. One works with r_λ for one suitably chosen λ. One finds a motive X over some finite Galois totally real extension F'/F which realizes the reduction $\overline{r}_\lambda$ in its mod l cohomology and whose mod l' cohomology is induced from a character. One tries to argue that by automorphic induction the mod l' cohomology is automorphic over F', hence by an automorphy lifting theorem the l'-adic cohomology is automorphic over F', hence tautologically the mod l cohomology is automorphic over F' and hence, finally, by another automorphy lifting theorem r_λ is automorphic over F'. To find X one uses a lemma of Moret-Bailly [MB89], [GPR95] and for this one needs a family of motives with distinct Hodge numbers, which has large monodromy. Griffiths transversality tells us that this will only be possible if the Hodge numbers of the motives are consecutive (e.g $0, 1, 2, \ldots, n-1$). Thus the l-adic cohomology of X may be automorphic of a different weight (infinitesimal character) than r_λ and the second automorphy lifting theorem needs to incorporate a 'change of weight'. In addition it seems that we can in general only expect to find X over an extension F'/F which is highly ramified at l. Thus our second automorphy lifting theorem needs to work over a base which is highly ramified at l. These two, related problems were the principal difficulties we faced. The original higher dimensional automorphy lifting theorems (see [CHT08], [Tay08]) could handle neither of them. In the ordinary case one of us (D.G.) proved an automorphy lifting theorem that uses Hida theory and some new local calculations to handle both of these problems (see [Ger09]). This has had important applications, but its applicability is still severely limited because we don't know how to prove that many compatible systems of l-adic representations are ordinary infinitely often.

The main innovation of this paper is a new automorphy lifting theorem that handles both these problems in significant generality. One of our key ideas is to introduce the the notion of a potentially crystalline representation ρ of the absolute Galois group of a local field K being *potentially diagonalizable*: ρ is potentially diagonalizable if there is a finite extension K'/K such that $\rho|_{G_{K'}}$ lies on the same irreducible component of the universal crystalline lifting ring of $\overline{\rho}|_{G_{K'}}$ (with fixed Hodge-Tate numbers) as a sum of characters lifting $\overline{\rho}|_{G_{K'}}$. (We remark that this does not depend on the choice of integral model for ρ.) Ordinary crystalline representations are potentially diagonalizable, as are crystalline representations in the Fontaine-Laffaille range (i.e. over an absolutely unramified base and with Hodge-Tate numbers in the range $[0, l-2]$). Potential diagonalizability is also preserved under restriction to the absolute Galois group of a finite extension. In this sense they behave better than 'crystalline representations in the Fontaine-Laffaille range' which require the ground field to be absolutely unramified. Finally 'potentially diagonalizable' representations are perfectly suited to our method of proving automorphy lifting theorems that allow for a change of weight. It seems to us to be a very interesting question to clarify further the ubiquity of potential diagonalizability. Could every crystalline representation be potentially diagonalizable? (We have no reason to believe this, but we know of no counterexample.)

The following gives an indication of the sort of automorphy lifting theorems we are able to prove. (See Theorem 4.2.1 and also section 2.1 for the definition of any notation or terminology which may be unfamiliar.)

Theorem B. *Let F be an imaginary CM field with maximal totally real subfield F^+ and let c denote the non-trivial element of $\mathrm{Gal}\,(F/F^+)$. Let n denote a positive integer. Suppose that $l \geq 2(n+1)$ is a prime such that F does not contain a primitive l^{th} root of 1. Let*

$$r : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation and let $\overline{r}$ denote the semi-simplification of the reduction of r. Also let

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

be a continuous character. Suppose that r and μ enjoy the following properties:

1. **(Odd essential conjugate-self-duality)** *$r^c \cong r^\vee \mu$ and $\mu(c_v) = -1$ for all $v|\infty$.*
2. **(Being unramified almost everywhere)** *r ramifies at only finitely many primes.*
3. **(Potential diagonalizability and regularity)** *$r|_{G_{F_v}}$ is potentially diagonalizable (and so in particular potentially crystalline) for all $v|l$ and for each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ it has n distinct τ-Hodge-Tate numbers.*
4. **(Irreducibility)** *The restriction $\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible.*
5. **(Residual ordinary automorphy)** *There is a RAECSDC automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*

   $$(\overline{r}, \overline{\mu}) \cong (\overline{r}_{l,\imath}(\pi), \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n})$$

   and π is $\imath$-ordinary.

Then (r, μ) is automorphic.

Theorem B implies the following potential automorphy theorem for a single l-adic representation, from which Theorem A can be deduced. (See Corollary 4.5.2 and Theorem 5.4.1.)

Theorem C. *Suppose that F is a totally real field. Let n be a positive integer and let $l \geq 2(n+1)$ be a prime. Let*

$$r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous representation. We will write $\overline{r}$ for the semi-simplification of the reduction of r. Suppose that the following conditions are satisfied.

1. **(Being unramified almost everywhere)** *r is unramified at all but finitely many primes.*
2. **(Odd essential self-duality)** *Either r maps to GSp_n with totally odd multiplier or it maps to $GO(n)$ with totally even multiplier.*
3. **(Potential diagonalizability and regularity)** *r is potentially diagonalizable (and hence potentially crystalline) at each prime v of F above l and for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ it has n distinct τ-Hodge-Tate numbers.*
4. **(Irreducibility)** *$\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible.*

Then we can find a finite Galois totally real extension F'/F such that $r|_{G_{F'}}$ is automorphic. Moreover r is part of a weakly compatible system of l-adic representations. (In fact, r is part of a strictly pure compatible system in the sense of section 5.1.)

This theorem has other applications besides Theorem A. For instance we mention the following irreducibility result (see Theorem 5.4.2).

Theorem D. *Suppose that F is a CM or totally real field and that π is a regular, algebraic, essentially conjugate self-dual, cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$. If π_∞ has sufficiently regular weight ('extremely regular' in the sense of section 2.1), then for l in a set of rational primes of Dirichlet density 1 the n-dimensional l-adic representations associated to π are irreducible.*

To prove Theorem B we employ Harris' tensor product trick (see [Har09]), which was first employed in connection with change of weight in [BLGG09]. However the freedom that 'potential diagonalizability' gives us to make highly ramified base changes in the non-ordinary case means that this method becomes more powerful. More precisely, suppose that r is potentially diagonalizable, and that r_0 is a potentially diagonalizable, automorphic lift of $\overline{r}$ (with possibly different Hodge-Tate numbers to r). In fact making a finite soluble base change we can assume they are diagonalizable, i.e. we can take $K' = K$ in the definition of potential diagonalizability. We choose a cyclic extension M/F of degree n in which each prime above l splits completely, and two characters θ and θ_0 of G_M such that

- $\bar{\theta} = \overline{\theta}_0$,
- the restriction of $\mathrm{Ind}\,_{G_M}^{G_F}\theta$ to an inertia group at a prime $v|l$ realizes a diagonal point on the same component of the universal crystalline lifting ring of $\overline{r}|_{G_{F_v}}$ as $r|_{G_{F_v}}$,
- and the restriction of $\mathrm{Ind}\,_{G_M}^{G_F}\theta_0$ to an inertia group at a prime $v|l$ realizes a diagonal point on the same component of the universal crystalline lifting ring of $\overline{r}|_{G_{F_v}}$ as $r_0|_{G_{F_v}}$.

Then $r_0 \otimes \mathrm{Ind}\,_{G_M}^{G_F}\theta$ is automorphic and has the same reduction as $r \otimes \mathrm{Ind}\,_{G_M}^{G_F}\theta_0$. Moreover the restrictions of these two representations to the decomposition group at a prime $v|l$ lie on the same component of the universal crystalline lifting ring of $(\overline{r} \otimes \mathrm{Ind}\,_{G_M}^{G_F}\overline{\theta}_0)|_{G_{F_v}}$. This is enough for the usual Taylor-Wiles-Kisin argument to prove that $r \otimes \mathrm{Ind}\,_{G_M}^{G_F}\theta_0$ is also automorphic, from which we can deduce (as in [BLGHT09]) the automorphy of r.

Things are a little more complicated than this because it seems to be hard to combine this with the 'level changing' argument in [Tay08]. In addition a direct argument imposes unwanted conditions on the Hodge-Tate numbers of r_0 and r. So instead of going directly from the automorphy of r_0 to that of r we create two ordinary lifts r_1 and r_2 of $\overline{r}$ (at least after a base change) where r_1 has the same local behaviour away from l as r_0; r_2 has the same local behaviour away from l as r; and where the Hodge-Tate numbers of r_1 and r_2 are chosen suitably. Our new arguments allow us to deduce the automorphy of r_1 from that of r_0. D.G.'s results in the ordinary case [Ger09] allow us to deduce the automorphy of r_2 from that of r_1. Finally applying our new argument again allows us to deduce the automorphy of r from the automorphy of r_2. To construct r_1 and r_2 we use the method of Khare and Wintenberger [KW09] based on potential automorphy (in the ordinary case, where it is already available: see for example [BLGHT09]).

Along the way we also prove a general theorem about the existence of l-adic lifts with prescribed local behaviour of a given mod l Galois representation (see Theorem 4.3.1). We deduce a rather general theorem about change of weight and level (see Theorem 4.4.1) of which a very particular instance is the following.

Theorem E. *Let n be a positive integer and let $l > 2(n+1)$ be a prime. Fix $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F be a CM field such that all primes of F above l are unramified over $\mathbb{Q}$ and split over the maximal totally real subfield of F. Let π be a regular, algebraic, essentially conjugate self-dual, cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ satisfying the following conditions:*

- *π is unramified above l;*
- *π_∞ has weight $(a_{\tau,i})_{\tau:F\hookrightarrow\mathbb{C},\, i=1,\dots,n}$ with $l-n-1 \geq a_{\tau,1} \geq a_{\tau,2} \geq \cdots \geq a_{\tau,n} \geq 0$ for all τ;*
- *and the restriction to $G_{F(\zeta_l)}$ of the mod l Galois representation $\overline{r}_{l,\imath}(\pi)$ associated to π and $\imath$ is irreducible.*

Note that in this case $a_{\tau,i} + a_{\tau c,n+1-i} = w$ is independent of τ and i. Suppose that we are given a second weight $(a'_{\tau,i})_{\tau:F\hookrightarrow\mathbb{C},\, i=1,\dots,n}$ with $l-n-1 \geq a'_{\tau,1} \geq a'_{\tau,2} \geq \cdots \geq a'_{\tau,n} \geq 0$ for all τ, such that

- *$a'_{\tau,i} + a'_{\tau c,n+1-i} = w$ is also independent of τ and i,*
- *and for all places $v|l$ of F the restriction $\overline{r}_{l,\imath}(\pi)|_{G_{F_v}}$ has a lift which is crystalline with τ-Hodge-Tate numbers $\{a'_{\imath\tau,i} + n - i\}$.*

Then there is a second regular, algebraic, essentially conjugate self-dual, cuspidal automorphic representation π' of $GL_n(\mathbb{A}_F)$ giving rise (via $\imath$) to the same mod l Galois representation (i.e. 'congruent to π mod l') such that π' is also unramified above l and π_∞ has weight $(a'_{\tau,i})$.

We now explain the structure of the paper. In section 1 we collect some results about the deformation theory of Galois representations. These are mostly now fairly standard results but we recall them to fix notations and in some cases to make slight improvements. The main exception is the introduction of potential diagonalizability in section 1.4, which is new and of key importance for us. In section 2 we fix some notations and we recall the existing automorphy lifting theorems (or slight generalizations of them). Very little in this section is novel. Between the writing of the first and second versions of this paper, Jack Thorne [Tho10] has found optimal versions of these theorems which allow one to remove the troublesome 'bigness' conditions from [CHT08] and the papers that followed it. Moreover Ana Caraiani [Car10] has proved local-global compatibility in all $l \neq p$ cases, as well as proving temperedness of all regular, algebraic, essentially conjugate self-dual cuspidal automorphic representations. We have taken advantage of Ana's and Jack's works to optimize our own results. In section 3 we make use of the automorphy lifting theorems from section 2 and the Dwork family to prove a potential automorphy theorem (in the ordinary case) and a theorem about lifting mod l Galois representations (again in the ordinary case). These arguments follow those of [BLGHT09] and will not surprise an expert.

In section 4 we prove our main new theorems. Section 4.1 contains our main new argument. In section 4.2 we combine this with the results of sections 2 and 3 to obtain our optimal automorphy lifting theorem. In section 4.3 we use the same ideas to deduce an improved result about the existence of l-adic lifts of mod l Galois representations with specified local behaviour. Combining the results of sections 4.2 and 4.3 we deduce in section 4.4 a general theorem about change of weight and level for mod l automorphic forms on GL_n. Then in section 4.5 we use the automorphy lifting theorem of section 4.2 and our potential automorphy theorem from section 3.3 to deduce our main new potential automorphy result for a single l-adic representation.

In section 5 we turn to applications of our main results. In section 5.1 we recall definitions connected to compatible systems of l-adic representations. In section 5.2 we prove some group theoretic lemmas about the images of compatible systems of l-adic representations. Then in section 5.3 we deduce from the potential automorphy theorem of section 4.5 our main theorem – a potential automorphy theorem for compatible systems of l-adic representations. Finally in section 5.4 we give further applications of our main results – applications to fitting an l-adic representation into a compatible system and to the irreducibility of some l-adic representations associated to cusp forms on $GL(n)$.

We are grateful to Ana Caraiani and Jack Thorne for sharing with us early drafts of their papers [Car10] and [Tho10] respectively. Our paper was written at the same time as [BLGG10] and there was considerable cross fertilization. Our paper would have been impossible without Harris' tensor product trick and it is a pleasure to acknowledge our debt to him.

Notation. We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. Let $\mathfrak{gl}_n$ denote the space of $n \times n$ matrices with the adjoint action of GL_n and let $\mathfrak{gl}_n^0$ denote the subspace of trace zero matrices. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R.

If Γ is a profinite group then Γ^{ab} will denote its maximal abelian quotient by a closed subgroup. If $\rho : \Gamma \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous homomorphism then we will let $\overline{\rho} : \Gamma \to GL_n(\overline{\mathbb{F}}_l)$ denote the *semi-simplification* of its reduction, which is well defined up to conjugacy.

If M is a field, we let $\overline{M}$ denote its algebraic closure and G_M denote its absolute Galois group. We will use ζ_n to denote a primitive n^{th}-root of 1. Let ϵ_l denote the l-adic cyclotomic character and $\overline{\epsilon}_l$ its reduction modulo l. We will also let $\omega_l : G_M \to \mu_{l-1} \subset \mathbb{Z}_l^\times$ denote the Teichmuller lift of $\overline{\epsilon}_l$. If N/M is a separable quadratic extension we will let $\delta_{N/M}$ denote the non-trivial character of $\mathrm{Gal}\,(N/M)$.

If K is a finite extension of $\mathbb{Q}_p$ for some p, we write K^{nr} for its maximal unramified extension; I_K for the inertia subgroup of G_K; $\mathrm{Frob}_K \in G_K/I_K$ for the geometric Frobenius; and W_K for the Weil group. We will write $\mathrm{Art}\,_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ for the Artin map normalized to send uniformizers to geometric Frobenius elements. We will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $GL_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Weil-Deligne representation of the Weil group W_K. We will write rec for rec_K when the choice of K is clear. If (r, N) is a Weil-Deligne representation of W_K we will write $(r, N)^{F-\mathrm{ss}}$ for its Frobenius semisimplification. If ρ is a continuous representation of G_K over $\overline{\mathbb{Q}}_l$ with $l \neq p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil-Deligne representation of W_K. (See for instance section 1 of [TY07].) By a *Steinberg* representation of $GL_n(K)$ we will mean a representation $\mathrm{Sp}\,_n(\psi)$ (in the notation of section 1.3 of [HT01]) where ψ is an unramified character of $K^\times$. If K'/K is a finite extension and if π is an irreducible smooth representation of $GL_n(K)$ we will write $\pi_{K'}$ for the base change of π to K' which is characterized by $\mathrm{rec}_{K'}(\pi_{K'}) = \mathrm{rec}_K(\pi)|_{W_{K'}}$.

If ρ is a continuous de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil-Deligne representation of W_K, and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ is a continuous embedding of fields then we will write $\mathrm{HT}_\tau(\rho)$ for the multiset of Hodge-Tate numbers of ρ with respect to τ. Thus $\mathrm{HT}_\tau(\rho)$ is a multiset of $\dim \rho$ integers. In fact if W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_l$ and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ then the multiset $\mathrm{HT}_\tau(W)$ contains i with multiplicity $\dim_{\overline{\mathbb{Q}}_l}(W \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\epsilon_l) = \{-1\}$.

If $K = \mathbb{R}$ or $\mathbb{C}$ or if K is a CM field, we will let c denote complex conjugation, a well defined automorphism of K. If K is a number field and $v|\infty$ is a place of K we will write Art_{K_v} for the unique isomorphism $K_v^\times/(K_v^\times)^0 \xrightarrow{\sim} \mathrm{Gal}\,(\overline{K}_v/K_v)$. If in addition v is real then we will let $[c_v]$ denote the conjugacy class in G_K consisting of complex conjugations associated to v.

We will write $|| \;\; ||$ for the continuous homomorphism

$$|| \;\; || = \prod_v | \;\; |_v : \mathbb{A}^\times/\mathbb{Q}^\times \longrightarrow \mathbb{R}^\times_{>0},$$

where each $| \;\; |_v$ has its usual normalization, i.e. $|p|_p = 1/p$. If $K/\mathbb{Q}$ is a finite extension we will write $|| \;\; ||_K$ (or simply $|| \;\; ||$) for $||\mathbf{N}_{K/\mathbb{Q}}\,||$. We will also write

$$\mathrm{Art}_K = \prod_v \mathrm{Art}_{K_v} : \mathbb{A}_K^\times/\overline{K^\times(K_\infty^\times)^0} \xrightarrow{\sim} G_K^{\mathrm{ab}}.$$

If v is a finite place of K we will write $k(v)$ for its residue field and Frob_v for Frob_{K_v}. If K'/K is a quadratic extension of number fields we will denote by $\delta_{K'/K}$ the nontrivial character of $\mathbb{A}_K^\times/K^\times \mathbf{N}_{K'/K}\mathbb{A}_{K'}^\times$. (We hope that this will cause no confusion with the Galois character $\delta_{K'/K}$. One equals the composition of the other with the Artin map for K.) If K'/K is a soluble, finite Galois extension and if π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_K)$ we will write $\pi_{K'}$ for its base change to K', an automorphic representation of $GL_n(\mathbb{A}_{K'})$.

Contents

Introduction.	2
1. Deformations of Galois Representations.	9
2. Automorphy Lifting Theorems I.	24
3. Potential Automorphy.	31
4. The main theorems.	38
5. Compatible systems.	51
References	67

1. Deformations of Galois Representations.

1.1. The group $\mathcal{G}_n$.

We let $\mathcal{G}_n$ denote the semi-direct product of $\mathcal{G}_n^0 = GL_n \times GL_1$ by the group $\{1, \jmath\}$ where

$$\jmath(g,a)\jmath^{-1} = (a^t g^{-1}, a).$$

We let $\nu : \mathcal{G}_n \to GL_1$ be the character which sends (g,a) to a and sends $\jmath$ to -1. We will also let $GSp_{2n} \subset GL_{2n}$ denote the symplectic similitude group defined by the anti-symmetric matrix

$$J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

and we will again let $\nu : GSp_{2n} \to GL_1$ denote the multiplier character. Finally let $GO(n)$ denote the orthogonal similitude group defined by the symmetric matrix 1_n.

There is a natural homomorphism

$$\mathcal{G}_n \times \mathcal{G}_m \longrightarrow \mathcal{G}_n/\mathcal{G}_n^0 \times \mathcal{G}_m/\mathcal{G}_m^0 \longrightarrow \{\pm 1\}$$

which sends both $(\jmath, 1)$ and $(1, \jmath)$ to -1. Let $(\mathcal{G}_n \times \mathcal{G}_m)^+$ denote the kernel of this map. There is a homomorphism

$$\begin{array}{rcl} \otimes : (\mathcal{G}_n \times \mathcal{G}_m)^+ & \longrightarrow & \mathcal{G}_{nm} \\ (g,a) \times (g',a') & \longmapsto & (g \otimes g', aa') \\ \jmath \times \jmath & \longmapsto & \jmath. \end{array}$$

There is also a homomorphism

$$\begin{array}{rcl} I : \mathcal{G}_n & \longrightarrow & GSp_{2n} \\ (g,a) & \longmapsto & \begin{pmatrix} g & 0 \\ 0 & a^t g^{-1} \end{pmatrix} \\ \jmath & \longmapsto & \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}. \end{array}$$

Suppose that Γ is a group with a normal subgroup Δ of index 2 and that $\gamma_0 \in \Gamma - \Delta$. Suppose also that A is a ring and that $r : \Gamma \to \mathcal{G}_n(A)$ is a homomorphism with $\Delta = r^{-1}\mathcal{G}_n^0(A)$. Write $\breve{r} : \Delta \to GL_n(A)$ for the composition of $r|_\Delta$ with projection to $GL_n(A)$. Write $r(\gamma_0) = (a, -(\nu \circ r)(\gamma_0))\jmath$. Then

$$\breve{r}(\gamma_0 \delta \gamma_0^{-1}) a^t \breve{r}(\delta) = (\nu \circ r)(\delta) a$$

for all $\delta \in \Delta$, and
$$\breve{r}(\gamma_0^2)^t a = -(\nu \circ r)(\gamma_0) a.$$

If $r : \Gamma \to GSp_{2n}(A)$ is a homomorphism with multiplier μ, then it gives rise to a homomorphism $\hat{r}_\Delta : \Gamma \to \mathcal{G}_{2n}(A)$ which sends $\delta \in \Delta$ to $(r(\delta), \mu(\delta))$ and $\gamma \in \Gamma - \Delta$ to $(r(\gamma) J_{2n}^{-1}, -\mu(\gamma))\jmath$. Then $\Delta = \hat{r}_\Delta^{-1} \mathcal{G}_{2n}^0(A)$ and $\nu \circ \hat{r}_\Delta = \mu$. Similarly if $r : \Gamma \to GO_n(A)$ is a homomorphism with multiplier μ, then it gives rise to a homomorphism $\hat{r}_\Delta : \Gamma \to \mathcal{G}_n(A)$ which sends $\delta \in \Delta$ to $(r(\delta), \mu(\delta))$ and $\gamma \in \Gamma - \Delta$ to $(r(\gamma), \mu(\gamma))\jmath$. Then $\Delta = \hat{r}_\Delta^{-1} \mathcal{G}_{2n}^0(A)$ and $\nu \circ \hat{r}_\Delta$ equals the product of μ with the nontrivial character of Γ/Δ.

If $r : \Gamma \to \mathcal{G}_n(A)$ (resp. $r' : \Gamma \to \mathcal{G}_m(A)$) is a homomorphism with $r^{-1}\mathcal{G}_n^0(A) = \Delta$ (resp. $(r')^{-1}\mathcal{G}_m^0(A) = \Delta$) then we define
$$I(r) = I \circ r : \Gamma \longrightarrow GSp_{2n}(A)$$
and
$$r \otimes r' = \otimes \circ (r \times r') : \Gamma \longrightarrow \mathcal{G}_{nm}(A).$$
Note that the multiplier of $I(r)$ equals the multiplier of r and that the multiplier of $r \otimes r'$ differs from the product of the multipliers of r and r' by the non-trivial character of Γ/Δ.

If $\chi : \Delta \to A^\times$ and $\mu : \Gamma \to A^\times$ satisfy

- $\chi \chi^{\gamma_0} = \mu|_\Delta$, and
- $\chi(\gamma_0^2) = -\mu(\gamma_0)$;

(i.e. the composition of χ with the transfer map $\Gamma^{\mathrm{ab}} \to \Delta^{\mathrm{ab}}$ equals the product of μ and the non-trivial character of Γ/Δ), then there is a homomorphism
$$\begin{array}{rcl} (\chi, \mu) : \Gamma & \longrightarrow & \mathcal{G}_1(A) \\ \delta & \longmapsto & (\chi(\delta), \mu(\delta)) \\ \gamma & \longmapsto & (\chi(\gamma\gamma_0^{-1}), -\mu(\gamma))\jmath, \end{array}$$
for all $\delta \in \Delta$ and $\gamma \in \Gamma - \Delta$. We have $\nu \circ (\chi, \mu) = \mu$.

In the case that $\Gamma = G_{F^+}$ and $\Delta = G_F$ where F is a CM field with maximal totally real subfield F^+, we call $r : G_{F^+} \to \mathcal{G}_n(A)$ (resp. $GSp_{2n}(A)$, resp. $GO_n(A)$) *odd* if the multiplier character takes every complex conjugation to -1 (resp. -1, resp. 1). Note that if r is odd so is $I(r)$ (resp. $\hat{r}_\Delta$, resp. $\hat{r}_\Delta$).

Suppose now that A is a field, that $r : \Delta \to GL_n(A)$ is absolutely irreducible, and that $\mu : \Gamma \to A^\times$ is a character so that
$$r^{\gamma_0} \cong r^\vee \otimes \mu|_\Delta.$$
More precisely if $\gamma \in \Gamma - \Delta$ there is a $b_\gamma \in GL_n(A)$, unique up to scalar multiples, such that
$$r(\gamma\delta\gamma^{-1}) b_\gamma {}^t r(\delta) = \mu(\delta) b_\gamma$$
for all $\delta \in \Delta$. Computing $r(\gamma^2\delta\gamma^{-2})$ in two ways and using the absolute irreducibility of r, we deduce that $r(\gamma^2)$ is a scalar multiple of $b_\gamma {}^t b_\gamma^{-1}$. Substituting $\delta = \gamma^2$ in the last displayed equation, we then deduce that
$$r(\gamma^2)^t b_\gamma = \pm\mu(\gamma) b_\gamma.$$
One can check that the sign in the above equation is independent of $\gamma \in \Gamma - \Delta$, and we will denote it $-\mathrm{sgn}\,(r, \mu)$. (To see this one uses the fact that one can take $b_{\delta\gamma} = r(\delta) b_\gamma$ for $\delta \in \Delta$ and $\gamma \in \Gamma - \Delta$.) Then we get a homomorphism
$$\tilde{r} : \Gamma \longrightarrow \mathcal{G}_n(A)$$

which sends $\delta \in \Delta$ to $(r(\delta), \mu(\delta))$ and sends γ_0 to $(b_{\gamma_0}, -\mathrm{sgn}\,(r,\mu)\mu(\gamma_0))\jmath$. In particular if $\mathrm{sgn}\,(r,\mu) = 1$ then $\nu \circ \tilde{r} = \mu$, while if $\mathrm{sgn}\,(r,\mu) = -1$ then $\mu^{-1}(\nu \circ \tilde{r})$ is the non-trivial character of Γ/Δ. Moreover $\breve{\tilde{r}} = r$.

1.2. **Abstract deformation theory.** Fix a rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. Let Γ denote a topologically finitely generated profinite group and let $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$ be a continuous homomorphism.

We will denote by

$$\rho^{\square} : \Gamma \to GL_n(R^{\square}_{\mathcal{O},\overline{\rho}})$$

the universal lifting of $\overline{\rho}$ to a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. (We impose no equivalence condition on lifts other than equality.) We will write

$$R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$$

for $R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$. Note that if $\mathcal{O} \subset \mathcal{O}'$ are two such rings of integers then $R^{\square}_{\mathcal{O},\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}' \cong R^{\square}_{\mathcal{O}',\overline{\rho}}$, and so, as the notation suggests, the ring $R^{\square}_{\overline{\rho}} \otimes \overline{\mathbb{Q}}_l$ does not depend on the choice of ring of integers $\mathcal{O}$.

The maximal ideals are dense in $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$. A prime ideal $\wp$ of $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ is maximal if and only if the residue field $k(\wp) = R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]_{\wp}/\wp$ is (topologically isomorphic to) a finite extension of L. (See for instance Lemma 2.6 of [Tay08]. For the 'if' part note that the image of $R^{\square}_{\mathcal{O},\overline{\rho}}$ in $k(\wp)$ is a compact $\mathcal{O}$-submodule of $k(\wp)$ with field of fractions $k(\wp)$. Thus $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l] \twoheadrightarrow k(\wp)$.) We get a continuous representation $\rho_{\wp} : \Gamma \to GL_n(k(\wp))$. The formal completion $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]^{\wedge}_{\wp}$ is the universal lifting ring for $\rho_{\wp}$, i.e. if A is an Artinian local $k(\wp)$-algebra with residue field $k(\wp)$ and if $\rho : \Gamma \to GL_n(A)$ is a continuous representation lifting $\rho_{\wp}$, then there is a unique continuous map of $k(\wp)$-algebras $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]^{\wedge}_{\wp} \to A$ so that $\rho^{\square}$ pushes forward to ρ. (Let R denote the image of $R^{\square}_{\mathcal{O},\overline{\rho}}$ in $A/\mathfrak{m}_A$. Let A^0 denote the R-subalgebra of A generated by the matrix entries of the image of ρ. Then A^0 is a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ and $\rho : \Gamma \to GL_n(A^0)$. The assertion follows easily.) In particular, if $H^2(\Gamma, \mathrm{ad}\,\rho_{\wp}) = (0)$ then $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ is formally smooth at $\wp$ of dimension

$$\dim_{k(\wp)} Z^1(\Gamma, \mathrm{ad}\,\rho_{\wp}) = n^2 + \dim_{k(\wp)} H^1(\Gamma, \mathrm{ad}\,\rho_{\wp}) - \dim_{k(\wp)} H^0(\Gamma, \mathrm{ad}\,\rho_{\wp})$$

(where we use continuous cohomology). (We learned these observations from Mark Kisin.)

Lemma 1.2.1. *Let H denote the subgroup of $GL_n(R^{\square}_{\mathcal{O},\overline{\rho}})$ consisting of elements which reduce modulo the maximal ideal to an element that centralizes the image of $\overline{\rho}$. Then H acts naturally on $R^{\square}_{\mathcal{O},\overline{\rho}}$ on the right. This action fixes each irreducible component of* $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$.

Proof. The action of $h \in H$ is via the map $R^{\square}_{\mathcal{O},\overline{\rho}} \to R^{\square}_{\mathcal{O},\overline{\rho}}$ along which the universal lifting ρ^{univ} pushes forward to $h\rho^{\mathrm{univ}}h^{-1}$.

Suppose that $h \in H$ and that $\mathcal{O}'$ is the ring of integers of a finite extension of L in $\overline{\mathbb{Q}}_l$, with maximal ideal λ'. Suppose that $\rho : \Gamma \to GL_n(\mathcal{O}')$ lifts $\overline{\rho}$. Then $h\rho h^{-1}$ also lifts $\overline{\rho}$. (We are using h both for an element of $GL_n(R^{\square}_{\mathcal{O},\overline{\rho}})$ and for its image in $GL_n(\mathcal{O}')$ under the map $R^{\square}_{\mathcal{O},\overline{\rho}} \to \mathcal{O}'$ induced by ρ.) Let $\mathcal{O}'\langle s,t\rangle$ denote the algebra of power series over $\mathcal{O}'$ with coefficients tending to zero. Set

$A = \mathcal{O}'\langle s,t\rangle/(s\det(t1_n+(1-t)h)-1)$, a complete topological domain with the λ'-adic topology. We have a continuous representation

$$\widetilde{\rho} = (t1_n+(1-t)h)\rho(t1_n+(1-t)h)^{-1} : \Gamma \longrightarrow GL_n(A).$$

Let A^0 denote the closed subalgebra of A generated by the the matrix entries of elements of the image of $\widetilde{\rho}$ and give it the subspace topology. Then

$$\widetilde{\rho} : \Gamma \longrightarrow GL_n(A^0),$$

and $\widetilde{\rho} \bmod \lambda' = \overline{\rho}$, and $\widetilde{\rho}$ pushes forward to ρ (resp. $h\rho h^{-1}$) under the continuous homomorphism $A^0 \to \mathcal{O}'$ induced by $t \mapsto 1$ (resp. $t \mapsto 0$). We will show that A^0 is a complete, noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. It will follow that there is a natural map $R^{\square}_{\mathcal{O},\overline{\rho}} \to A^0$ through which the maps $R^{\square}_{\mathcal{O},\overline{\rho}} \to \mathcal{O}'$ corresponding to ρ and $h\rho h^{-1}$ both factor. As A^0 is a domain (being a sub-ring of A) we conclude that the points corresponding to ρ and $h\rho h^{-1}$ lie on the same irreducible component of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$. As any irreducible component contains an $\mathcal{O}'$-point which lies on no other irreducible component for some $\mathcal{O}'$ as above (because such points are Zariski dense in $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$), we see that the lemma follows.

It remains to show that A^0 is a complete, noetherian, local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. Let $\gamma_1,\dots,\gamma_r$ denote topological generators of Γ. Write

$$\widetilde{\rho}(\gamma_i) = a_i + b_i,$$

where $a_i \in GL_n(\mathcal{O})$ lifts $\overline{\rho}(\gamma_i)$ and where $b_i \in M_{n\times n}(\lambda' A)$. Then A^0 is the closure of the $\mathcal{O}$-subalgebra of A generated by the entries of the b_i. As these entries are topologically nilpotent in A we get a continuous $\mathcal{O}$-algebra homomorphism

$$\mathcal{O}[[X_{ijk}]]_{i=1,\dots,r;\ j,k=1,\dots,n} \longrightarrow A$$

which sends X_{ijk} to the (j,k)-entry of b_i. Let J denote the kernel. As $\mathcal{O}[[X_{ijk}]]/J$ is compact (and A is Hausdorff) this map is a topological isomorphism of $\mathcal{O}[[X_{ijk}]]/J$ with its image in A and this image is closed. Thus the image is just A^0 and we have $\mathcal{O}[[X_{ijk}]]/J \xrightarrow{\sim} A^0$, so that A^0 is indeed a complete, noetherian, local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. $\square$

Now suppose that

$$\overline{r} : \Gamma \longrightarrow \mathcal{G}_n(\mathbb{F})$$

is a continuous homomorphism such that $\breve{\overline{r}}$ is absolutely irreducible and $\Gamma \twoheadrightarrow \mathcal{G}_n/\mathcal{G}_n^0$. Let Δ denote the kernel of $\Gamma \twoheadrightarrow \mathcal{G}_n/\mathcal{G}_n^0$. Then there is a universal deformation

$$r^{\mathrm{univ}} : \Gamma \longrightarrow \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{O},\overline{r}})$$

to a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$, where now we consider two liftings as equivalent deformations if they are conjugate. (See section 2.2 of [CHT08].)

Lemma 1.2.2. (1) *Suppose that $\Sigma \subset \Gamma$ has finite index, that Σ is not contained in Δ and that $\breve{\overline{r}}|_{\Delta\cap\Sigma}$ is absolutely irreducible. Then the natural map $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}|_\Sigma} \to R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ induced by $r^{\mathrm{univ}}|_\Sigma$ makes $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ a finitely generated $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}|_\Sigma}$-module.*

(2) *Suppose that $s : \Gamma \to \mathcal{G}_m(\mathcal{O})$ is a continuous homomorphism such that $\Delta = s^{-1}\mathcal{G}^0_m(\mathcal{O})$ and $\breve{\overline{s}} \otimes \breve{\overline{r}}$ is absolutely irreducible. Then the natural map $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}\otimes\overline{s}} \to R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ induced by $r^{\mathrm{univ}} \otimes s$ makes $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ a finitely generated $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}\otimes\overline{s}}$-module.*

(3) *Suppose $I(\overline{r})$ is absolutely irreducible and that Σ is another open subgroup of index two in Γ which does not contain* $\ker I(\overline{r})$. *Then the natural map $R^{\mathrm{univ}}_{\mathcal{O},\widehat{I(\overline{r})}_\Sigma} \to R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ induced by $\widehat{I(r^{\mathrm{univ}})}_\Sigma$ makes $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ a finitely generated $R^{\mathrm{univ}}_{\mathcal{O},\widehat{I(\overline{r})}_\Sigma}$-module.*

Proof. This is essentially Lemma 3.2.1 of [BLGG10]. Write R for $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}|_\Sigma}$ resp. $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}\otimes\overline{s}}$ resp. $R^{\mathrm{univ}}_{\mathcal{O},\widehat{I(\overline{r})}_\Sigma}$ and write $\mathfrak{m}$ for the maximal ideal of R. In each case we can check that the image of Γ in $\mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}})$ is finite. Let m denote the order of this image, and let $\gamma_1,\ldots,\gamma_m \in \Gamma$ be chosen so that their images in $\mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}})$ exhaust the image of Γ. Let

$$f(T) = \prod_{(\zeta_1,\ldots,\zeta_n)\in\mu_m(\overline{\mathbb{F}})^n} (T-(\zeta_1+\cdots+\zeta_n)) \in \mathbb{F}[T]$$

and let A denote the maximal quotient of $\mathbb{F}[X_{i,j}]_{i,j=1,\ldots,n}$ over which the m^{th}-power of the matrix $(X_{i,j})$ is 1_n. If $\wp$ is a prime ideal of A then all the roots of the characteristic polynomial of $(X_{i,j})$ over $A_\wp/\wp$ are m^{th} roots of unity and hence $f(\mathrm{tr}\,(X_{i,j})) = 0$ in $A/\wp \subset A_\wp/\wp$. Thus there is a positive integer a such that $f(\mathrm{tr}\,(X_{i,j}))^a = 0$ in A. Then we get a map

$$\begin{array}{rcl} \mathbb{F}[T_1,\ldots,T_m]/(f(T_1)^a,\ldots,f(T_m)^a) & \longrightarrow & R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}} \\ T_i & \longmapsto & \mathrm{tr}\,\breve{r}^{\mathrm{univ}}(\gamma_i). \end{array}$$

By Lemma 2.1.12 of [CHT08] we see that this map has dense image. On the other hand the source has finite cardinality. We conclude that the map is surjective and that $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}/\mathfrak{m}R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ is finite over $\mathbb{F}$. Hence by Nakayama's Lemma we conclude that that $R^{\mathrm{univ}}_{\mathcal{O},\overline{r}}$ is finite over R, as desired. □

1.3. **Local theory:** $l \neq p$.

Continue to fix a rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. However in this section we specialize our discussion to the case $\Gamma = G_K$, where $K/\mathbb{Q}_p$ is a finite extension and $p \neq l$. Thus $\overline{\rho} : G_K \to GL_n(\mathbb{F})$ is continuous. Write q for the order of the residue field of K.

In this case the local Euler characteristic formula tells us that the tangent space to $R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ at a maximal ideal $\wp$ has dimension $n^2 + \dim_{k(\wp)} H^2(G_K, \mathrm{ad}\,\rho_\wp)$. Note also that by local duality $\dim_{k(\wp)} H^2(G_K,\mathrm{ad}\,\rho_\wp) = \dim_{k(\wp)} H^0(G_K,(\mathrm{ad}\,\rho_\wp)(1))$. The closed points $\wp$ of $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ with $H^0(G_K,(\mathrm{ad}\,\rho_\wp)(1)) = (0)$ are Zariski dense (Theorem 2.1.6 of [Gee10], or see [Cho09]) and so all components of $R^{\square}_{\mathcal{O},\rho}[1/l]$ have dimension n^2 and are generically formally smooth. We will need a slight strengthening of this. We will call a continuous representation $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_l)$ *robustly smooth* (resp. *smooth*) if $H^0(G_{K'},(\mathrm{ad}\,\rho_\wp)(1)) = (0)$ for all finite extensions K'/K (resp. for $K' = K$). We will show below that the set of closed points of $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ which are robustly smooth are also Zariski dense.

Define a partial order on $\overline{L}^\times$ by $a \geq b$ if a equals $\sigma(b)\zeta q^m$ where $\sigma \in \mathrm{Gal}\,(\overline{L}/L)$, where ζ is a root of unity and where $m \in \mathbb{Z}_{\geq 0}$. We will write $a \approx b$ if $a \geq b$ and $b \geq a$; and we will write $a \sim b$ if $a \geq b$ or $b \geq a$. Further we will write $a > b$ for $a \geq b$ but $a \not\approx b$. Choose $\phi \in W_K$ a lift of Frob_K. If V is a finite dimensional L-vector space with an action of W_K with open kernel and if $a \in \overline{L}^\times$ then we define $V((a))$ (resp. $V(a)$) to be the L-subspaces of V such that $V((a)) \otimes_L \overline{L}$ (resp. $V(a) \otimes_L \overline{L}$) is the sum of the b-generalized eigenspaces of ϕ in V as b runs over all elements of $\overline{L}^\times$ with $a \approx b$ (resp. $a \sim b$). This is independent of the choice of ϕ. (If ϕ' is another choice then the actions of ϕ^m and $(\phi')^m$ on V are equal for some $m \in \mathbb{Z}_{>0}$.) Thus $V(a)$ and $V((a))$ are W_K-invariant. We have decompositions

$$V = \bigoplus V((a))$$

where a runs over $\overline{L}^\times / \approx$, and

$$V = \bigoplus V(a)$$

where a runs over $\overline{L}^\times / \sim$. We will say that V has type a if $V = V((a))$.

Lemma 1.3.1. *Suppose that (r, N) is a Weil-Deligne representation of W_K on a finite dimensional L-vector space V. Then we can write*

$$V = \bigoplus_{i=1}^{u} \bigoplus_{j=1}^{s_i} V_{i,j}$$

where

- *$V_{i,j}$ is invariant under I_K;*
- *$N : V_{i,j} \xrightarrow{\sim} V_{i,j+1}$ unless $j = s_i$ in which case $NV_{i,s_i} = (0)$;*
- *$W_K V_{i,j} \subset V_{i,j} \oplus \bigoplus_{i'=1}^{i-1} \bigoplus_{j'} V_{i',j'}$ and so we get an induced action of W_K on $V_{i,j}$;*
- *the action of W_K on $(V_{i,j} \oplus \bigoplus_{i'=1}^{i-1} \bigoplus_{j'} V_{i',j'})/(\bigoplus_{i'=1}^{i-1} \bigoplus_{j'} V_{i',j'})$ is irreducible;*
- *$V_{i,j}$ has type $a_i q^{1-j}$ for some $a_i \in \overline{\mathbb{Q}}_l^\times$;*
- *and if $i' < i$ then $a_i \not> a_{i'}$.*

Proof. We may suppose that $V = V(b)$ for some b (because N must take any $V(b)$ to itself). We will construct the $V_{i,j}$ by recursion on i. Suppose that we have constructed $V_{i,j}$ for $i < t$. Choose $a_t \in \overline{L}^\times$ such that

$$\left(V / \bigoplus_{i=1}^{t-1} \bigoplus_{j} V_{i,j} \right) ((a_t)) \neq (0)$$

and such that if $a > a_t$ then

$$\left(V / \bigoplus_{i=1}^{t-1} \bigoplus_{j} V_{i,j} \right) ((a)) = (0)$$

Also choose an irreducible W_K-submodule

$$\overline{V}_{t,1} \subset \left(V / \bigoplus_{i=1}^{t-1} \bigoplus_{j} V_{i,j} \right) ((a_t))$$

and choose s_t minimal such that $N^{s_t} \overline{V}_{t,1} = (0)$. Lift $\overline{V}_{t,1}$ to an I_K-submodule $V_{t,1}^0 \subset V((a_t))$. Then $N^{s_t} V_{t,1}^0 \subset \bigoplus_{i=1}^{t-1} \bigoplus_j V_{i,j}$. For each $i < t$ choose $j_i \in \mathbb{Z}_{>s_t}$

such that $a_t q^{-s_t} \approx a_i q^{1-j_i}$. (To see that $j_i > s_t$ we are using the fact that for $i \leq t$ we have $a_i \geq a_t$.) Then

$$N^{s_t} V_{t,1}^0 \subset \bigoplus_{i<t} V_{i,j_i}.$$

Thus if $v \in V_{t,1}^0$ we can write

$$N^{s_t} v = \sum_{i<t} N^{s_t} v_i$$

for unique elements $v_i \in V_{i,j_i-s_t}$. Set $V_{t,1}$ to be the set of

$$v - \sum_{i<t} v_i.$$

We see that $V_{t,1} \subset V((a_t))$ is a $\overline{\mathbb{Q}}_l$-sub-vector space lifting $\overline{V}_{t,1}$, which is I_K-invariant and satisfies $N^{s_t} V_{t,1} = (0)$. Set $V_{t,j} = N^{j-1} V_{t,1}$. It is not hard to see that these $V_{i,j}$ have all the desired properties. □

We remark that if we define an increasing filtration on V by

$$\mathrm{Fil}_i V = \bigoplus_{i' \leq i} \bigoplus_j V_{i,j}$$

then

$$V^{F-\mathrm{ss}} \cong \bigoplus_{i=1}^{u} \mathrm{gr}_i V.$$

Lemma 1.3.2. (1) *Suppose $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous representation; that $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and that π is an irreducible smooth representation of $GL_n(K)$ over $\mathbb{C}$ with $\imath\mathrm{WD}(\rho)^{F-\mathrm{ss}} \cong \mathrm{rec}_K(\pi)$. If π is generic then ρ is smooth.*

(2) *The closed points of $\wp$ in $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$ for which $\rho_\wp$ is robustly smooth are Zariski dense.*

Proof. For the first part write $\pi = \mathrm{Sp}_{s_1}(\pi_1) \boxplus \cdots \boxplus \mathrm{Sp}_{s_t}(\pi_t)$ for some supercuspidal representations π_i of $\mathrm{GL}_{n_i}(K)$ and positive integers s_i, n_i with $\sum s_i n_i = n$. (We are using the notation of [HT01].) Then ρ has a filtration with graded pieces ρ_i satisfying $\imath\mathrm{WD}(\rho_i)^{F-\mathrm{ss}} = \mathrm{rec}_K(\mathrm{Sp}_{s_i}(\pi_i))$, possibly after reordering the i's. Thus $(\mathrm{ad}\,\rho)(1)$ has a filtration with graded pieces $\mathrm{Hom}\,(\rho_i, \rho_j(1))$. If this had non-zero invariants, then $\pi_i \cong \pi_j \otimes |\det|^m$ for some $\max\{1, 1+s_j-s_i\} \leq m \leq s_j$. Thus (π_i, s_i) and (π_j, s_j) are linked contradicting the fact that π is generic (see page 36 of [HT01]).

For the second part suppose that $\wp$ is a closed point of $\mathrm{Spec}\, R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]$. Set $\mathcal{O}'$ equal to the image of $R^{\square}_{\mathcal{O},\overline{\rho}}$ in $L' = R^{\square}_{\mathcal{O},\overline{\rho}}[1/l]/\wp$ and let λ' denote the maximal ideal of $\mathcal{O}'$. By Lemma 1.3.1 we can find a decomposition $(L')^n = \bigoplus_{i=1}^u V_i$ such that

- for each i the sub-space V_i is invariant under I_K;
- for each i the sub-space $\mathrm{Fil}_i V = \bigoplus_{i'=1}^i V_{i'}$ is invariant under G_K;
- and for each i we have $\mathrm{WD}(\mathrm{gr}_i V) = \mathrm{Sp}_{s_i}(W_i)$, where W_i has some type a_i.

(By $\mathrm{Sp}_s(W)$ we mean the Weil-Deligne representation of W_K whose underlying representation of W_K is $W \oplus W(1) \oplus \cdots \oplus W(s-1)$ and where $N : W(i) \xrightarrow{\sim} W(i+1)$ for $i = 0, \ldots, s-2$.) Choose $M \in \mathbb{Z}_{>0}$ so that

$$\bigoplus_{i=1}^{u} ((\mathcal{O}')^n \cap V_i) \supset (l^{M-1}\mathcal{O}')^n.$$

Let

$$A \in \ker(GL_n(\mathcal{O}'[[X_1, \ldots, X_u]]) \longrightarrow GL_n(\mathbb{F}))$$

be the unique element which preserves each $(V_i \cap (\mathcal{O}')^n) \otimes_{\mathcal{O}'} \mathcal{O}'[[X_1, \ldots, X_u]]$ and acts on it by multiplication by $(1 + l^M X_i)$. Note that A commutes with $\rho_\wp(I_K)$. Then there is a unique continuous representation

$$\rho : G_K \longrightarrow GL_n(\mathcal{O}'[[X_1, \ldots, X_u]])$$

such that $\rho|_{I_K} = \rho_\wp|_{I_K} \otimes_{\mathcal{O}'} \mathcal{O}'[[X_1, \ldots, X_u]]$ and such that for any lift ϕ of Frob_K to W_K we have $\rho(\phi) = \rho_\wp(\phi)A$. Then ρ is a lift of $\overline{\rho}$. If $x \in (\lambda')^u$ write ρ_x for $\rho \bmod (X_1 - x_1, \ldots, X_u - x_u)$. Note that $\rho_0 = \rho_\wp$. We will show that for (Zariski) generic x that ρ_x is robustly smooth and the second part of the lemma will follow. (Note that if $0 \neq f \in \mathcal{O}'[[X_1, ..., X_u]]$ then f can not vanish on all of $(\lambda')^u$.)

If $y \in (\mathcal{O}')^\times$ let $\nu_y : G_K/I_K \to (\mathcal{O}')^\times$ be the unramified character taking Frob_K to y. Then $(\mathrm{ad}\, \rho_x)(1)$ has a filtration with graded pieces

$$\mathrm{Hom}\,(V_i, V_j(\nu_{(1+l^M x_j)/q(1+l^M x_i)})).$$

Note that if $i = j$ then

$$\mathrm{Hom}_{G_{K'}}(V_i, V_i(\nu_{1/q})) = (0)$$

for any finite K'/K, because

$$\mathrm{Hom}_{W_{K'}}(W_i, W_i(\nu_{1/q^j})) = (0)$$

for $j = 1, \ldots, s_j + 1$ (because, in turn, W_i and $W_i(\nu_{1/q^j})$ will have different types). So it remains to show that for general x we will also have

$$\mathrm{Hom}_{G_{K'}}(V_i, V_j(\nu_{(1+l^M x_j)/q(1+l^M x_i)})) = (0)$$

for all $i \neq j$ and all finite K'/K. Let $\phi \in W_K$ denote a Frobenius lift and let L'' denote the compositum of all extensions of L' of degree less than or equal to n. Then L''/L' is finite. It will do to choose $(x_i) \in (\lambda')^u$ so that if $i \neq j$ and if α (resp. β) is an eigenvalue of ϕ on V_i (resp. V_j) and if ζ is a root of unity then $q\alpha\zeta(1+l^M x_i) \neq \beta(1+l^M x_j)$. However if such an equality were to hold then $\zeta \in L''$. As L'' contains only finitely many roots of unity, the x_i's need only satisfy finitely many inequalities, as desired. $\square$

(This gives a third proof that $R^\square_{\mathcal{O},\overline{\rho}}[1/l]$ is generically formally smooth of dimension n^2, which seems to be different from those in [Gee10] and [Cho09].)

Suppose that $\mathcal{C}$ is a set of irreducible components of $\mathrm{Spec}\, R^\square_{\mathcal{O},\overline{\rho}}[1/l]$ and let $R^\square_{\mathcal{O},\overline{\rho},\mathcal{C}}$ denote the maximal quotient of $R^\square_{\mathcal{O},\overline{\rho}}$, which is reduced, l-torsion free and has $\mathrm{Spec}\, R^\square_{\mathcal{O},\overline{\rho},\mathcal{C}}[1/l]$ supported on the components in $\mathcal{C}$. Also let $\mathcal{D}_\mathcal{C}$ denote the set of liftings of $\overline{\rho}$ to complete local noetherian $\mathcal{O}$-algebras R with residue field $\mathbb{F}$ such that the induced map $R^\square_{\mathcal{O},\overline{\rho}} \to R$ factors through $R^\square_{\mathcal{O},\overline{\rho},\mathcal{C}}$. By Lemma 1.2.1 above and Lemma 3.2 of [BLGHT09] we see that $\mathcal{D}_\mathcal{C}$ is a deformation problem in the sense of Definition 2.2.2 of [CHT08].

If K'/K is finite and Galois we will let $R^{\square}_{\mathcal{O},\overline{\rho},K'-\mathrm{nr}}$ denote the maximal quotient of $R^{\square}_{\mathcal{O},\overline{\rho}}$ over which $\rho^{\square}(I_{K'})=\{1_n\}$. Then $R^{\square}_{\mathcal{O},\overline{\rho},K'-\mathrm{nr}}[1/l]$ is formally smooth of dimension n^2. (One again identifies the formal completions at closed points as universal deformation rings and uses the fact that $H^2(G_K/I_{K'},W)=(0)$ for any finite dimensional L-vector space W with continuous $G_K/I_{K'}$-action.) Thus $R^{\square}_{\mathcal{O},\overline{\rho},K'-\mathrm{nr}}[1/l] = R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}_{K'-\mathrm{nr}}}[1/l]$ for some finite set of components $\mathcal{C}_{K'-\mathrm{nr}}$ of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho}}$. Let $\mathcal{C}_{\text{p-nr}}$ denote the union of $\mathcal{C}_{K'-\mathrm{nr}}$ over all finite Galois extensions K'/K and set $R^{\square}_{\mathcal{O},\overline{\rho},\text{p-nr}} = R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}_{\text{p-nr}}}$. A $\overline{\mathbb{Q}}_l$-point of $R^{\square}_{\mathcal{O},\overline{\rho}}$ factors through $R^{\square}_{\mathcal{O},\overline{\rho},\text{p-nr}}$ if and only if it is potentially unramified. The ring $R^{\square}_{\mathcal{O},\overline{\rho},\text{p-nr}}$ is formally smooth of dimension n^2.

For $i=1,2$, let

$$\rho_i : G_K \longrightarrow GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

be a continuous representation. We say that ρ_1 *connects to* ρ_2, which we denote $\rho_1 \sim \rho_2$, if and only if

- the reduction $\overline{\rho}_1 = \rho_1 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$ is equivalent to the reduction $\overline{\rho}_2 = \rho_2 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$, and
- ρ_1 and ρ_2 define points on a common irreducible component of $\operatorname{Spec}(R^{\square}_{\overline{\rho}_1} \otimes \overline{\mathbb{Q}}_l)$.

We say that ρ_1 *strongly connects to* ρ_2, which we write $\rho_1 \rightsquigarrow \rho_2$, if $\rho_1 \sim \rho_2$ and ρ_1 lies on a unique irreducible component of $\operatorname{Spec}(R^{\square}_{\overline{\rho}_1} \otimes \overline{\mathbb{Q}}_l)$.

We make the following remarks.

(1) By Lemma 1.2.1 the relations $\rho_1 \sim \rho_2$ and $\rho_1 \rightsquigarrow \rho_2$ do not depend on the equivalence chosen between the reductions $\overline{\rho}_1$ and $\overline{\rho}_2$, nor on the $GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugacy class of ρ_1 or ρ_2.
(2) 'Connects' is a symmetric relationship, but 'strongly connects' may not be.
(3) 'Strongly connects' is a transitive relationship, whereas 'connects' may not be.
(4) If $\rho_1 \sim \rho_2$ and $\rho_2 \rightsquigarrow \rho_3$ then $\rho_1 \sim \rho_3$.
(5) If $\rho_1 \sim \rho_2$ and $H^0(G_K, (\operatorname{ad}\rho_1)(1)) = (0)$ then $\rho_1 \rightsquigarrow \rho_2$.
(6) Write $\mathrm{WD}(\rho_i) = (r_i, N_i)$. If $\rho_1 \sim \rho_2$ then $r_1|_{I_K} \cong r_2|_{I_K}$. If $\rho_1 \rightsquigarrow \rho_2$ and $\rho_2 \rightsquigarrow \rho_1$ then $(r_1|_{I_K}, N_1) \cong (r_2|_{I_K}, N_2)$. (See [Cho09].)
(7) If ρ_1 and ρ_2 are unramified and have the same reduction then $\rho_1 \sim \rho_2$.
(8) If K'/K is a finite extension and $\rho_1 \sim \rho_2$ then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$.
(9) If $\rho_1 \sim \rho_2$ and $\rho_1' \sim \rho_2'$ then $\rho_1 \oplus \rho_1' \sim \rho_2 \oplus \rho_2'$ and $\rho_1 \otimes \rho_1' \sim \rho_2 \otimes \rho_2'$ and $\rho_1^\vee \sim \rho_2^\vee$.
(10) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous character and if $\rho_1 \rightsquigarrow \rho_2$ then $\rho_1^\vee \rightsquigarrow \rho_2^\vee$ and $\rho_1 \otimes \mu \rightsquigarrow \rho_2 \otimes \mu$.
(11) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous unramified character with $\overline{\mu} = 1$ then $\rho_1 \sim \rho_1 \otimes \mu$.
(12) Suppose that $\overline{\rho}_1$ is semisimple and let Fil^i be an invariant decreasing filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$-direct summands, then $\rho_1 \sim \bigoplus_i \operatorname{gr}^i \rho_1$.

[We sketch the proof of the last of these assertions. We may suppose that L is chosen large enough that $\rho_1 : G_K \to GL_n(\mathcal{O})$ and that Fil^i is defined over L. Then we may choose a basis $\{e_{i,j}\}$ of $\mathcal{O}^n$ such that

- for each i the set $\{e_{i',j} :\ i \geq i'\}$ is a basis of $\operatorname{Fil}^i \mathcal{O}^n$;

- and for each i the set $\{e_{i,j}\}$ (with only j varying) spans a G_K submodule of $\mathbb{F}^n$.

(Use reverse induction on i.) Let $\mathcal{O}\langle t\rangle$ denote the algebra of power series over $\mathcal{O}$ with coefficients tending to 0, so that $\mathcal{O}\langle t\rangle$ is complete in the l-adic topology. Let h denote the element of $M_{n\times n}(\mathcal{O}\langle t\rangle)$ such that $he_{i,j} = t^i e_{i,j}$ for all i and j. Consider the continuous representation

$$\rho = h\rho_1 h^{-1} : G_K \longrightarrow GL_n(\mathcal{O}\langle t\rangle).$$

Let A^0 denote the closed subalgebra of $\mathcal{O}\langle t\rangle$ generated by the matrix entires of the image of ρ. As in the proof of Lemma 1.2.1, we see that A^0 is a complete, noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ and that there is a continuous homomorphism $R^\square_{\mathcal{O},\overline{\rho}_1} \to A^0$ under which the universal lifting of $\overline{\rho}_1$ pushes forward to ρ. Under the map $A^0 \to \mathcal{O}$ which sends t to 1, we see that ρ pushes forward to ρ_1. Under the map $A^0 \to \mathcal{O}$ which sends t to 0, we see that ρ pushes forward to $\bigoplus_i \mathrm{gr}^i \rho_1$. As A^0 is a domain, the claim follows.]

Important convention: Suppose that F is a global field and that $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous representation with irreducible reduction $\overline{r}$. In this case there is a model $r^\circ : G_F \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of r, which is unique up to $GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugation. If $v|p$ is a place of F we write $r|_{G_{F_v}} \sim \rho_2$ (resp. $r|_{G_{F_v}} \rightsquigarrow \rho_2$, resp. $\rho_1 \rightsquigarrow r|_{G_{F_v}}$) to mean $r^\circ|_{G_{F_v}} \sim \rho_2$ (resp. $r^\circ|_{G_{F_v}} \rightsquigarrow \rho_2$, resp. $\rho_1 \rightsquigarrow r^\circ|_{G_{F_v}}$).

1.4. **Local theory:** $l = p$.

Continue to fix a rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. However in this section we specialize our discussion to the case $\Gamma = G_K$, where $K/\mathbb{Q}_l$ is a finite extension. Thus $\overline{\rho} : G_K \to GL_n(\mathbb{F})$ is continuous. We will assume that the image of each continuous embedding $K \hookrightarrow \overline{L}$ is contained in L. Let $\{H_\tau\}$ be a collection of n element multisets of integers parametrized by $\tau \in \mathrm{Hom}_{\mathbb{Q}_l}(K, \overline{\mathbb{Q}}_l)$.

We call a continuous representation $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_l)$ *ordinary* if the following conditions are satisfied:

- there is a G_K-invariant decreasing filtration Fil^i on $\overline{\mathbb{Q}}_l^n$ such that for $i = 1, \ldots, n$ the graded piece $\mathrm{gr}^i \overline{\mathbb{Q}}_l^n$ is one dimensional and G_K acts on it by a character χ_i;
- and there are integers $b_{\tau,i} \in \mathbb{Z}$ for $\tau \in \mathrm{Hom}_{\mathbb{Q}_l}(K, \overline{\mathbb{Q}}_l)$ and $i = 1, \ldots, n$ and an open subgroup $U \subset K^\times$ such that
  - $(\chi_i \circ \mathrm{Art}_K)|_U(\alpha) = \prod_{\tau : K \hookrightarrow \overline{\mathbb{Q}}_l} (\tau\alpha)^{b_{\tau,i}}$
  - and $b_{\tau,1} < b_{\tau,2} < \cdots < b_{\tau,n}$ for all τ.

We will call ρ *ss-ordinary* if we may take $U = \mathcal{O}_K^\times$ in the above definition. We will call ρ *cr-ordinary* if it is ordinary and crystalline (in which case it is also ss-ordinary). If ρ is ordinary (resp. ss-ordinary) then it is de Rham (resp. semi-stable) and

$$\mathrm{HT}_\tau(\rho) = \{-b_{\tau,1}, \ldots, -b_{\tau,n}\}.$$

If ρ is ss-ordinary and if for each i there exists a τ such that $b_{\tau,i} + 1 < b_{\tau,i+1}$ then ρ is cr-ordinary. (See Propositions 1.24, 1.26 and 1.28 of [Nek93] or Lemma 3.1.4 of [GG09].)

Let K'/K denote a finite extension. The universal lifting ring $R^\square_{\mathcal{O},\overline{\rho}}$ has various important quotients $R^\square_{\mathcal{O},\overline{\rho},\{H_\tau\},*}$ which are uniquely characterized by requiring that they are reduced without l-torsion and that a $\overline{\mathbb{Q}}_l$-point of $R^{\square}_{\mathcal{O},\overline{\rho}}$ factors through $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*}$ if and only if it corresponds to a representation $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_l)$ which is de Rham with Hodge-Tate numbers $\mathrm{HT}_\tau(\rho) = H_\tau$ for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ and which has a further specified property $\mathcal{P}_*$. We will consider the following instances of this construction:

- $* =$ cris and $\mathcal{P}_*$ is 'crystalline';
- $* =$ ss and $\mathcal{P}_*$ is 'semi-stable';
- $* = K' -$ cris and $\mathcal{P}_*$ is 'crystalline after restriction to $G_{K'}$';
- $* = K' -$ ss and $\mathcal{P}_*$ is 'semi-stable after restriction to $G_{K'}$';
- $* =$ cr-ord and $\mathcal{P}_*$ is 'cr-ordinary';
- $* =$ ss-ord and $\mathcal{P}_*$ is 'ss-ordinary'.

We will write $R^{\square}_{\overline{\rho},\{H_\tau\},*} \otimes \overline{\mathbb{Q}}_l$ for $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$. This definition is independent of the choice of $\mathcal{O}$.

If H_τ has n distinct elements for each τ then each ring $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},*}$ is either zero or equidimensional of dimension

$$1 + n^2 + [K : \mathbb{Q}_l]n(n-1)/2.$$

If $K'' \supset K'$ then $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{cris}}$ (resp. $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{ss}}$) is a union of irreducible components of $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K''-\mathrm{cris}}$ (resp. $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K''-\mathrm{ss}}$). Each of the schemes $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}[1/l]$ and $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{cris}}[1/l]$ and $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cr\text{-}ord}}[1/l]$ are formally smooth. Finally if $\overline{\rho}$ is trivial then the scheme $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cr\text{-}ord}}[1/l]$ is geometrically irreducible. (In the cases $* = \mathrm{cris}, \mathrm{ss}, K'-\mathrm{cris}, K'-\mathrm{ss}$ this all follows from [Kis08]. In the case $* = \mathrm{cr\text{-}ord}, \mathrm{ss\text{-}ord}$ it follows from Lemmas 3.3.3 and 3.4.3 of [Ger09].)

Choose a finite set $\mathcal{C}$ of irreducible components of $\lim_{\to K'} \operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{ss}}$. Let $R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$ denote the maximal quotient of of $R^{\square}_{\mathcal{O},\overline{\rho},\{H_\tau\},K'-\mathrm{ss}}$ which is reduced, l-torsion free and has $\operatorname{Spec} R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$ supported on the components in $\mathcal{C}$, for K' chosen sufficiently large. This is independent of the choice of K' (as long as K' is sufficiently large). Also let $\mathcal{D}_{\mathcal{C}}$ denote the set of liftings of $\overline{\rho}$ to complete local noetherian $\mathcal{O}$-algebras R with residue field $\mathbb{F}$ such that the induced map $R^{\square}_{\mathcal{O},\overline{\rho}} \to R$ factors through $R^{\square}_{\mathcal{O},\overline{\rho},\mathcal{C}}$. Again we see that $\mathcal{D}_{\mathcal{C}}$ is a deformation problem in the sense of definition 2.2.2 of [CHT08].

If ρ_1 and ρ_2 are continuous representations $G_K \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ are continuous representations, we say that ρ_1 *connects to* ρ_2, which we denote $\rho_1 \sim \rho_2$, if and only if

- the reduction $\overline{\rho}_1 = \rho_1 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$ is equivalent to the reduction $\overline{\rho}_2 = \rho_2 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$;
- ρ_1 and ρ_2 are both potentially crystalline;
- for each continuous $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\rho_1) = \mathrm{HT}_\tau(\rho_2)$;
- and ρ_1 and ρ_2 define points on the same irreducible component of the scheme $\operatorname{Spec}(R^{\square}_{\overline{\rho}_1,\{\mathrm{HT}_\tau(\rho_1)\},K'-\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l)$ for some (and hence all) sufficiently large K'.

We make the following remarks.

(1) By the proof of lemma 1.2.1 we see that the relation $\rho_1 \sim \rho_2$ does not depend on the equivalence chosen between the reductions $\overline{\rho}_1$ and $\overline{\rho}_2$, nor on the $GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugacy class of ρ_1 or ρ_2.
(2) 'Connects' is an equivalence relation. (Because $R^{\square}_{\mathcal{O},\overline{\rho}_i,\{H_\tau\},\text{cr-ord}}[1/l]$ is formally smooth.)
(3) If $\rho_1 \sim \rho_2$ then $\mathrm{WD}(\rho_1)|_{I_K} \cong \mathrm{WD}(\rho_2)|_{I_K}$. (See [Kis08].)
(4) If K'/K is a finite extension and $\rho_1 \sim \rho_2$ then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$.
(5) If $\rho_1 \sim \rho_2$ and $\rho_1' \sim \rho_2'$ then $\rho_1 \oplus \rho_1' \sim \rho_2 \oplus \rho_2'$ and $\rho_1 \otimes \rho_1' \sim \rho_2 \otimes \rho_2'$ and $\rho_1^\vee \sim \rho_2^\vee$.
(6) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous unramified character with $\overline{\mu} = 1$ and ρ_1 is potentially crystalline then $\rho_1 \sim \rho_1 \otimes \mu$.
(7) Suppose that ρ_1 is potentially crystalline and that $\overline{\rho}_1$ is semisimple. Let Fil^i be an invariant filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$ direct summands, then $\rho_1 \sim \bigoplus_i \mathrm{gr}^i \rho_1$. (This is proved in the same way as remark 12 of the previous section.)

We will call a representation $\rho : G_K \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonalizable* if it is crystalline and connects to some representation $\chi_1 \oplus \cdots \oplus \chi_n$ with $\chi_i : G_K \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$ crystalline characters. We will call a representation $\rho : G_K \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *potentially diagonalizable* if there is a finite extension K'/K such that $\rho|_{G_{K'}}$ is diagonalizable. Note that if K''/K is a finite extension and ρ is diagonalizable (resp. potentially diagonalizable) then $\rho|_{G_{K''}}$ is diagonalizable (resp. potentially diagonalizable). It seems to us an interesting, and important, question to determine which potentially crystalline representations are potentially diagonalizable. As far as we know they could all be.

Lemma 1.4.1. *If ρ_1 and ρ_2 are conjugate in $GL_n(\overline{\mathbb{Q}}_l)$ then ρ_1 is potentially diagonalizable if and only if ρ_2 is potentially diagonalizable.*

Thus we can speak of a representation $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_l)$ being potentially diagonalizable without needing to specify an invariant lattice.

Proof. If ρ_1 and ρ_2 are conjugate by an element of $GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ then after passing to a finite extension over which $\overline{\rho}_1 = \overline{\rho}_2 = 1$ we see that $\rho_1 \sim \rho_2$. Thus we may suppose that $\rho_1 = g\rho_2 g^{-1}$ where $g = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i \in \overline{\mathbb{Q}}_l^\times$ satisfying $d_n | d_{n-1} | \ldots | d_1$. Choosing $L \subset \overline{\mathbb{Q}}_l$ large enough we may assume that ρ_1 and ρ_2 are defined over $\mathcal{O}$ and that $d_1, \ldots, d_n \in \mathcal{O}$. (If d_1 is not integral multiply each d_i by a suitable element of $L^\times$.) Replacing K by a finite extension we may also assume that $\rho_2 \equiv 1 \bmod l d_1/d_n$, in which case we also have $\overline{\rho}_1 = 1$.

Consider the complete topological domain

$$A = \mathcal{O}\langle t_1, s_1, t_2, s_2, \ldots, t_{n-1}, s_{n-1}\rangle / (s_1 t_1 - (d_1/d_2), \ldots, s_{n-1}t_{n-1} - (d_{n-1}/d_n)).$$

Let $\widetilde{g} = \mathrm{diag}(t_1 \cdots t_{n-1}, t_2 \cdots t_{n-1}, \ldots, t_{n-1}, 1)$ and let

$$\widetilde{\rho} = \widetilde{g}\rho_2 \widetilde{g}^{-1}.$$

If $j \geq i$ then the (i,j) entry of $\widetilde{\rho}(\sigma)$ is $t_i \ldots t_{j-1}$ times the (i,j) entry of $\rho_2(\sigma)$. If $i \geq j$ then the (i,j) entry of $\widetilde{\rho}(\sigma)$ is $s_j \ldots s_{i-1} d_i/d_j$ times the (i,j) entry of $\rho_2(\sigma)$. Thus we see that $\widetilde{\rho} : G_K \to GL_n(A)$ is a continuous homomorphism. The specialization under $t_i \mapsto 1$ for all i is ρ_2. The specialization under $s_i \mapsto 1$ for all i is ρ_1. As in the proof of lemma 1.2.1 we conclude that $\rho_1 \sim \rho_2$, and we are done. $\square$

We will establish some cases of (potential) diagonalizability below, but first we must recall some results from the theory of Fontaine and Laffaille [FL82], normalized as in section 2.4.1 of [CHT08]. Assume that $K/\mathbb{Q}_l$ is unramified and denote its ring of integers by $\mathcal{O}_K$. Let $\mathcal{MF}_{\mathcal{O}}$ denote the category of finite $\mathcal{O}_K \otimes_{\mathbb{Z}_l} \mathcal{O}$-modules M together with

- a decreasing filtration $\mathrm{Fil}^i M$ by $\mathcal{O}_K \otimes_{\mathbb{Z}_l} \mathcal{O}$-submodules which are $\mathcal{O}_K$-direct summands with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{l-1} M = \{0\}$;
- and $\mathrm{Frob}_p^{-1} \otimes 1$-linear maps $\Phi^i : \mathrm{Fil}^i M \to M$ with $\Phi^i|_{\mathrm{Fil}^{i+1} M} = l\Phi^{i+1}$ and $\sum_i \Phi^i \mathrm{Fil}^i M = M$.

Let $\mathrm{REP}_{\mathcal{O}}(G_K)$ denote the category of finite $\mathcal{O}$-modules with a continuous G_K-action. There is an exact, fully faithful, covariant functor of $\mathcal{O}$-linear categories $\mathbf{G}_K : \mathcal{MF}_{\mathcal{O}} \to \mathrm{REP}_{\mathcal{O}}(G_K)$. The essential image of $\mathbf{G}_K$ is closed under taking sub-objects and quotients. If M is an object of $\mathcal{MF}_{\mathcal{O}}$, then the length of M as an $\mathcal{O}$-module is $[K : \mathbb{Q}_l]$ times the length of $\mathbf{G}_K(M)$ as an $\mathcal{O}$-module.

Let $\mathbb{F}$ denote the residue field of $\mathcal{O}$ and let $\mathcal{MF}_{\mathbb{F}}$ denote the full subcategory of $\mathcal{MF}_{\mathcal{O}}$ consisting of objects killed by the maximal ideal λ of $\mathcal{O}$ and let $\mathrm{REP}_{\mathbb{F}}(G_K)$ denote the category of finite $\mathbb{F}$-modules with a continuous G_K-action. Then $\mathbf{G}_K$ restricts to a functor $\mathcal{MF}_{\mathbb{F}} \to \mathrm{REP}_{\mathbb{F}}(G_K)$. If M is an object of $\mathcal{MF}_{\mathbb{F}}$ and τ is a continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_l$, we let $\mathrm{FL}_\tau(M)$ denote the multiset of integers i such that $\mathrm{gr}^i M \otimes_{\mathcal{O}_K \otimes_{\mathbb{Z}_l} \mathcal{O}, \tau \otimes 1} \mathcal{O} \neq \{0\}$ and i is counted with multiplicity equal to the $\mathbb{F}$-dimension of this space. If M is an l-torsion free object of $\mathcal{MF}_{\mathcal{O}}$ then $\mathbf{G}_K(M) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is crystalline and for every continuous embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ we have

$$\mathrm{HT}_\tau(\mathbf{G}_K(M) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = \mathrm{FL}_\tau(M \otimes_{\mathcal{O}} \mathbb{F}).$$

Moreover, if Λ is a G_K-invariant lattice in a crystalline representation V of G_K with all its Hodge-Tate numbers in the range $[0, l-2]$ then Λ is in the image of $\mathbf{G}_K$. (See [FL82].)

Lemma 1.4.2. *Let $K/\mathbb{Q}_l$ be unramified. Let $\overline{M}$ denote an object of $\mathcal{MF}_{\mathbb{F}}$ together with a filtration*

$$\overline{M} = \overline{M}_0 \supset \overline{M}_1 \supset \cdots \supset \overline{M}_{n-1} \supset \overline{M}_n = (0)$$

by $\mathcal{MF}_{\mathbb{F}}$-subobjects such that $\overline{M}_i/\overline{M}_{i+1}$ has $\mathbb{F}$-rank $[K : \mathbb{Q}_l]$ for $i = 0, \ldots, n-1$. Then we can find an object M of $\mathcal{MF}_{\mathcal{O}}$ which is l-torsion free together with a filtration by $\mathcal{MF}_{\mathcal{O}}$-subobjects

$$M = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = (0)$$

and an isomorphism

$$M \otimes_{\mathcal{O}} \mathbb{F} \cong \overline{M}$$

under which $M_i \otimes_{\mathcal{O}} \mathbb{F}$ maps isomorphically to $\overline{M}_i$ for all i.

Proof. $\overline{M}$ has an $\mathbb{F}$ basis $\overline{e}_{i,\tau}$ for $i = 1, \ldots, n$ and $\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$ such that

- the residue field k_K of K acts on $\overline{e}_{i,\tau}$ via τ;
- $\overline{M}_j$ is spanned over $\mathbb{F}$ by the $\overline{e}_{i,\tau}$ for $i > j$;
- and for each j there is a subset $\Omega_j \subset \{1, \ldots, n\} \times \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$ such that $\mathrm{Fil}^j \overline{M}$ is spanned over $\mathbb{F}$ by the $\overline{e}_{i,\tau}$ for $(i, \tau) \in \Omega_j$.

Then we define M to be the free $\mathcal{O}$-module with basis $e_{i,\tau}$ for $i = 1, \ldots, n$ and $\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$.

- We let $\mathcal{O}_K$ act on $e_{i,\tau}$ via τ;
- we define M_j to the sub $\mathcal{O}$-module generated by the $e_{i,\tau}$ with $i > j$;
- and we define $\mathrm{Fil}^j M$ to be the $\mathcal{O}$-submodule spanned by the $e_{i,\tau}$ for $(i,\tau) \in \Omega_j$.

We define $\Phi^j : \mathrm{Fil}^j M \to M$ by reverse induction on j. If we have defined Φ^{j+1} we define Φ^j as follows:

- If $(i,\tau) \in \Omega_{j+1}$ then $\Phi^j e_{i,\tau} = l\Phi^{j+1} e_{i,\tau}$.
- If $(i,\tau) \in \Omega_j - \Omega_{j+1}$ then $\Phi^j e_{i,\tau}$ is chosen to be an $\mathcal{O}$-linear combination of the $e_{i',\tau\circ\mathrm{Frob}_p}$ for $i' \geq i$ which lifts $\overline{\Phi}^j \overline{e}_{i,\tau}$.

It follows from Nakayama's lemma that M is an object of $\mathcal{MF}_{\mathcal{O}}$, and then it is easy to verify that it has the desired properties. □

We can now state and prove our potential diagonalizability criteria.

Lemma 1.4.3. *Keep the above notation, including the assumption $l = p$. Suppose that $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_l)$ is a potentially crystalline representation.*

1. *If ρ has a G_K-invariant filtration with one dimensional graded pieces, in particular if it is ordinary, then ρ is potentially diagonalizable.*
2. *If $K/\mathbb{Q}_l$ is unramified, if ρ is crystalline and if for each $\tau : K \hookrightarrow \overline{L}$ the Hodge-Tate numbers $\mathrm{HT}_\tau(\rho_1) \subset [a_\tau, a_\tau + l - 2]$ for some integer a_τ, then ρ_1 is potentially diagonalizable.*

Proof: After passing to a finite extension so that $\overline{\rho}$ becomes trivial and each $\mathrm{gr}^i \rho$ becomes crystalline, the first part follows from item (7) of the first numbered list of this section.

For the second we may assume (by twisting) that $a_\tau = 0$ for all τ. Note that every irreducible subquotient of $\overline{\rho}|_{I_K}$ is trivial on wild inertia and hence one dimensional. Choose a finite unramified extension K'/K such that $\overline{\rho}(G_{K'}) = \overline{\rho}(I_K)$. Then $\overline{\rho}|_{G_{K'}}$ has a $G_{K'}$ invariant filtration with 1-dimensional graded pieces. From Lemma 1.4.2 (and the discussion just proceeding it) we see that $\overline{\rho}|_{G_{K'}}$ has a crystalline lift ρ_2 with the same Hodge-Tate numbers as $\rho|_{G_{K'}}$ which also has a $G_{K'}$-invariant filtration with one dimensional graded pieces. It follows from Lemma 2.4.1 of [CHT08] that $\rho|_{G_{K'}} \sim \rho_2$. (Note that in section 2.4.1 of [CHT08] there is a running assumption that the Hodge-Tate numbers are distinct. However this assumption is not used in the proof of Lemma 2.4.1 of [CHT08].) From the first part of this lemma we see that ρ_2 is potentially diagonalizable. Hence ρ is also potentially diagonalizable. □

Important convention: Suppose that F is a global field and that $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous representation with irreducible reduction $\overline{r}$. In this case there is model $r^\circ : G_F \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of r, which is unique up to $GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugation. If $v|l$ is a place of F we write $r|_{G_{F_v}} \sim \rho_2$ to mean $r^\circ|_{G_{F_v}} \sim \rho_2$. We will also say that $r|_{G_{F_v}}$ is (potentially) diagonalizable to mean that $r^\circ|_{G_{F_v}}$ is.

1.5. **Global theory.**

Fix an odd rational prime l and let $\mathcal{O}$ denote the ring of integers of a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\mathbb{F} = \mathcal{O}/\lambda$. Let F denote an imaginary CM field with maximal totally real subfield F^+. We suppose that each prime of F^+ above l splits in F and that L contains the image of each embedding $F \hookrightarrow \overline{L}$. Let S denote a finite set of places of F^+ which split in F and suppose that S contains all places of F^+ above l. For each $v \in S$ choose once and for all a prime $\widetilde{v}$ of F above v, and let $\widetilde{S}$ denote the set of $\widetilde{v}$ for $v \in S$.

Let

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\mathbb{F})$$

be a continuous representation such that $G_F = \overline{r}^{-1}\mathcal{G}_n^0(\mathbb{F})$ and such that $\overline{r}$ is unramified outside S. Let

$$\mu : G_{F^+} \longrightarrow \mathcal{O}^\times$$

be a continuous character lifting $\nu \circ \overline{r}$. We suppose that μ is de Rham, so that there is an integer w such that $\mathrm{HT}_\tau(\mu) = \{w\}$ for all $\tau : F^+ \hookrightarrow L$. For $\tau : F \hookrightarrow L$ let H_τ denote a multiset of n integers such that

$$H_{\tau \circ c} = \{w - h : \ h \in H_\tau\}.$$

For $v \in S$ with $v \not| l$ choose a set $\mathcal{C}_v$ of irreducible components of the scheme $\operatorname{Spec} R^{\square}_{\mathcal{O}, \breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}}[1/l]$, and let $\mathcal{D}_v$ denote the corresponding local deformation problem. For $v \in S$ with $v|l$ choose a finite set $\mathcal{C}_v$ of irreducible components of $\lim_{\rightarrow K'} \operatorname{Spec} R^{\square}_{\mathcal{O}, \breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}, \{H_\tau\}, K'-\mathrm{ss}}$, and let $\mathcal{D}_v$ denote the corresponding local deformation problem.

Let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \mu, \{\mathcal{D}_v\}_{v \in S}).$$

The following proposition is established in [CHT08] (see Proposition 2.2.9 and Corollary 2.3.5 of that paper).

Proposition 1.5.1. *Keep the notation and assumptions established in this section. Suppose moreover that $\breve{\overline{r}}$ is absolutely irreducible.*

(1) *There is a universal deformation*

$$r^{\mathrm{univ}}_{\mathcal{S}} : G_{F^+} \longrightarrow \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}})$$

of $\overline{r}$ of type $\mathcal{S}$ in the sense of section 2.3 of [CHT08].

(2) *If $\mu(c_v) = -1$ for all $v|\infty$ (i.e. if μ is totally odd) and if each H_τ has n distinct elements then $R^{\mathrm{univ}}_{\mathcal{S}}$ has Krull dimension at least* 1.

2. Automorphy Lifting Theorems I.

2.1. **Terminology.**

Continue to fix a rational prime l and an isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$.

Suppose that F is a CM (or totally real) field and that F^+ is the maximal totally real subfield of F. Let

$$r : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

(resp. $\overline{r} : G_F \longrightarrow GL_n(\overline{\mathbb{F}}_l)$) and

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

(resp. $\overline{\mu} : G_{F^+} \longrightarrow \overline{\mathbb{F}}_l^\times$) be continuous homomorphisms. We will call the pair (r, μ) (resp. $(\overline{r}, \overline{\mu})$) *essentially conjugate self-dual* if for some infinite place v of F^+ there is $\varepsilon_v \in \{\pm 1\}$ and a non-degenerate pairing $\langle\ ,\ \rangle_v$ on $\overline{\mathbb{Q}}_l^n$ (resp. $\overline{\mathbb{F}}_l^n$) such that

$$\langle x, y\rangle_v = \varepsilon_v \langle y, x\rangle_v$$

and

$$\langle r(\sigma)x, r(c_v\sigma c_v)y\rangle_v = \mu(\sigma)\langle x, y\rangle_v$$

(resp.

$$\langle \overline{r}(\sigma)x, \overline{r}(c_v\sigma c_v)y\rangle_v = \overline{\mu}(\sigma)\langle x, y\rangle_v)$$

for all $x, y \in \overline{\mathbb{Q}}_l^n$ (resp. $\overline{\mathbb{F}}_l^n$) and all $\sigma \in G_F$. In the case that F is imaginary we further require that $\varepsilon_v = -\mu(c_v)$. (This last condition can always be achieved by replacing μ by $\mu\delta_{F/F^+}$.)

Note the following.

- If the condition is true for one place $v|\infty$ it will be true for all places $v|\infty$: take $\varepsilon_{v'} = \mu(c_v c_{v'})\varepsilon_v$ and $\langle x, y\rangle_{v'} = \langle x, r(c_v c_{v'})y\rangle_v$ (resp. $\langle x, y\rangle_{v'} = \langle x, \overline{r}(c_v c_{v'})y\rangle_v$).
- If F is imaginary then (r, μ) (resp. $(\overline{r}, \overline{\mu})$) is essentially conjugate self-dual if and only if there is a continuous homomorphism $\widetilde{r} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$ (resp. $\mathcal{G}_n(\overline{\mathbb{F}}_l)$) with $\breve{\widetilde{r}} = r$ (resp. $\overline{r}$) and with multiplier μ (resp. $\overline{\mu}$).
- If F is totally real then (r, μ) is essentially conjugate self-dual if and only if r factors through $GSp_n(\overline{\mathbb{Q}}_l)$ (if $\mu(c_v) = -\varepsilon_v$) or $GO_n(\overline{\mathbb{Q}}_l)$ (if $\mu(c_v) = \varepsilon_v$) with multiplier μ. [Define the pairing on $\overline{\mathbb{Q}}_l^n$ by $\langle x, y\rangle = \langle x, r(c_v)y\rangle_v$.] A similar assertion is true in the case of $\overline{\mathbb{F}}_l$ if $l > 2$.

We will call (r, μ) (resp. $(\overline{r}, \overline{\mu})$) *totally odd, essentially conjugate self-dual* if they are essentially conjugate self-dual and $\varepsilon_v = 1$ for all $v|\infty$. (Or what comes to the same thing if $\mu(c_v)$ (resp. $\overline{\mu}(c_v)$) is independent of $v|\infty$ and $\varepsilon_v = 1$ for some $v|\infty$.)

Suppose that F is a number field and

$$\chi : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$$

is a continuous character. If there exists $a \in \mathbb{Z}^{\mathrm{Hom}\,(F,\mathbb{C})}$ such that

$$\chi|_{(F_\infty^\times)^0} : x \longmapsto \prod_{\tau \in \mathrm{Hom}\,(F,\mathbb{C})} (\tau x)^{a_\tau}$$

we will call χ algebraic. In that case we can associate to χ a de Rham, continuous character

$$r_{l,\imath}(\chi) : G_F \longrightarrow \overline{\mathbb{Q}}_l^\times$$

as at the start of section 1 of [BLGHT09]. If $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ recall that $\mathrm{HT}_\tau(r_{l,\imath}(\chi)) = \{-a_{\imath\circ\tau}\}$.

We now recall from [CHT08] and [BLGHT09] the notions of RAESDC and RAECSDC automorphic representations. In fact, it will be convenient for us to work with a slight variant of these definitions, where we keep track of the character which occurs in the essential (conjugate) self-duality.

Let F be an imaginary CM field with maximal totally real subfield F^+. By a *RAECSDC* (regular, algebraic, essentially conjugate self dual, cuspidal) automorphic representation of $GL_n(\mathbb{A}_F)$ we mean a pair (π, χ) where

- π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ such that π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n,
- $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ is a continuous character such that $\chi_v(-1) = (-1)^n$ for all $v|\infty$,
- and $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$.

We will say that (π, χ) has *level prime to l* (resp. *level potentially prime to l*) if for all $v|l$ the representation π_v is unramified (resp. becomes unramified after a finite base change).

Now let F be a totally real field. By a *RAESDC* (regular, algebraic, essentially self dual, cuspidal) automorphic representation of $GL_n(\mathbb{A}_F)$ we mean a pair (π, χ) where

- π is a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ such that π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n,
- $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ is a continuous character such that $\chi_v(-1)$ is independent of $v|\infty$,
- and $\pi \cong \pi^\vee \otimes (\chi \circ \det)$.

We will say that (π, χ) has *level prime to l* (resp. *level potentially prime to l*) if for all $v|l$ the representation π_v is unramified (resp. becomes unramified after a finite base change).

If Ω is an algebraically closed field of characteristic 0 we will write $(\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\Omega),+}$ for the set of $a = (a_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\Omega)}$ satisfying

$$a_{\tau,1} \geq \cdots \geq a_{\tau,n}.$$

Let $w \in \mathbb{Z}$. If F is totally real or CM (resp. if $\Omega = \mathbb{C}$) we will write $(\mathbb{Z}^n)_w^{\mathrm{Hom}\,(F,\Omega)}$ for the subset of elements $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\Omega)}$ with

$$a_{\tau,i} + a_{\tau\circ c,n+1-i} = w$$

(resp.

$$a_{\tau,i} + a_{c\circ\tau,n+1-i} = w.)$$

(These definitions are consistent when F is totally real or CM and $\Omega = \mathbb{C}$.) If F'/F is a finite extension we define $a_{F'} \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F',\Omega),+}$ by

$$(a_{F'})_{\tau,i} = a_{\tau|_F,i}.$$

We will call a *extremely regular* if for some τ the $a_{\tau,i}$ have the following property: for any subsets H and H' of $\{a_{\tau,i} + n - i\}_{i=1}^n$ of the same cardinality, if $\sum_{h\in H} h = \sum_{h\in H'} h$ then $H = H'$.

If $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\mathbb{C}),+}$, let Ξ_a denote the irreducible algebraic representation of $GL_n^{\mathrm{Hom}\,(F,\mathbb{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a RAESDC (resp. RAECSDC) automorphic representation (π,χ) of $GL_n(\mathbb{A}_F)$ has *weight* a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$. Note that in this case a must lie in $(\mathbb{Z}^n)_w^{\mathrm{Hom}\,(F,\mathbb{C})}$ for some $w \in \mathbb{Z}$.

We recall that to a RAESDC or RAECSDC representation (π,χ) of $GL_n(\mathbb{A}_F)$ we can attach a continuous semi-simple representation

$$r_{l,\imath}(\pi) : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

with the properties described in Theorem 1.1 (resp. 1.2) of [BLGHT09]. The pair $(r_{l,\imath}(\pi), \epsilon_l^{1-n} r_{l,\imath}(\chi))$ is totally odd, essentially conjugate self-dual. (See Theorem 1.2 and Corollary 1.3 of [BC09]. Note that Theorem 1.2 of [BC09] can easily be extended to the case χ non-trivial by a twisting argument. Also note that irreducible factors r of $r_{l,\imath}(\pi)$ which do not satisfy $r^c \cong r^\vee \otimes \epsilon_l^{1-n} r_{l,\imath}(\chi)$ occur in pairs $r, (r^c)^\vee \otimes \epsilon_l^{1-n} r_{l,\imath}(\chi)$ and it is straightforward to put a pairing of the desired form on $r \oplus (r^c)^\vee \otimes \epsilon_l^{1-n} r_{l,\imath}(\chi)$.) When F is CM, note that by definition $\epsilon_l^{1-n} r_{l,\imath}(\chi)$ takes every complex conjugation to -1, i.e. is totally odd.

The representation $r_{l,\imath}(\pi)$ is de Rham (crystalline if π_v is unramified for all $v|l$) and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ then

$$\mathrm{HT}_\tau(r_{l,\imath}(\pi)) = \{a_{\imath\tau,1} + n - 1, a_{\imath\tau,2} + n - 2, \ldots, a_{\imath\tau,n}\}.$$

If $v \not| l$ then Caraiani's local-global compatibility tells us that

$$\imath \mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2}).$$

(See [Car10].)

We will let $\overline{r}_{l,\imath}(\pi)$ denote the semisimplification of the reduction of $r_{l,\imath}(\pi)$. If F is totally real and if $\chi_v(-1) = (-1)^{n-1}$ for all $v|\infty$ then $\overline{r}_{l,\imath}(\pi)$ factors through a map

$$\widetilde{\overline{r}}_{l,\imath}(\pi) : G_F \longrightarrow GO_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\epsilon}_l^{1-n} \overline{r}_{l,\imath}(\chi)$. If F is totally real and if $\chi_v(-1) = (-1)^n$ for all $v|\infty$ then n is even and $\overline{r}_{l,\imath}(\pi)$ factors through a map

$$\widetilde{\overline{r}}_{l,\imath}(\pi) : G_F \longrightarrow GSp_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\epsilon}_l^{1-n} \overline{r}_{l,\imath}(\chi)$. If F is imaginary CM then it extends to a continuous homomorphism

$$\widetilde{\overline{r}}_{l,\imath}(\pi) : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\epsilon}_l^{1-n} \overline{r}_{l,\imath}(\chi)$.

In definition 5.1.2 of [Ger09] D.G. defines what it means for a regular algebraic cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$ to be *$\imath$-ordinary*. For our purposes the exact definition will not be so important, rather all that will matter are the following facts. We let (π,χ) denote a RAESDC or RAECSDC automorphic representation of $GL_n(\mathbb{A}_F)$.

(1) If π is $\imath$-ordinary then $r_{l,\imath}(\pi)$ is ordinary.
(2) If ψ is an algebraic character of $\mathbb{A}_F^\times/F^\times$ then π is $\imath$-ordinary if and only if $\pi \otimes (\psi \circ \det)$ is $\imath$-ordinary. (This follows directly from the definition.)
(3) If π has weight 0 and if π_v is Steinberg for all $v|l$ then π is $\imath$-ordinary. (See Lemma 5.1.5 of [Ger09].)

(4) If π has level potentially prime to l and if $r_{l,\imath}(\pi)$ is ordinary then π is $\imath$-ordinary. (See Lemmas 5.1.6 and 5.2.1 of [Ger09])

We remark that it is presumably both true and provable that π is $\imath$-ordinary if and only if $r_{l,\imath}(\pi)$ is ordinary, but to work out the details here would take us too far afield.

We will call a pair (r,μ) (resp. $(\overline{r},\overline{\mu})$) consisting of a Galois representation and an algebraic Galois character *automorphic* if there is a RAESDC or RAECSDC representation (π,χ) such that

$$(r,\mu)\cong(r_{l,\imath}(\pi),r_{l,\imath}(\chi)\epsilon_l^{1-n})$$

(resp.

$$(\overline{r},\overline{\mu})\cong(\overline{r}_{l,\imath}(\pi),\overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n})).$$

We will say that (r,μ) or $(\overline{r},\overline{\mu})$ is *automorphic of level prime to l* (resp. *automorphic of level potentially prime to l*, resp. *ordinarily automorphic*) if (π,χ) has level prime to l (resp. has level potentially prime to l, resp. is $\imath$-ordinary).

The next Lemma formalizes the argument of step 3 of the proof of Theorem 7.5 of [BLGHT09].

Lemma 2.1.1. *Let F be a totally real field or CM field and M/F a cyclic imaginary Galois CM extension of degree m. Let $r:G_M\to GL_n(\overline{\mathbb{Q}}_l)$ be a continuous representation. Assume that there is a RAESDC automorphic representation (Π,χ) of $GL_{mn}(\mathbb{A}_F)$ and $\imath:\overline{\mathbb{Q}}_l\xrightarrow{\sim}\mathbb{C}$ such that $r_{l,\imath}(\Pi)\cong\mathrm{Ind}_{G_M}^{G_F}r$. Then there is a RAECSDC automorphic representation $(\pi,(||\;||_F^{n-nm}\chi)\circ\mathbf{N}_{M^+/F})$ of $GL_n(\mathbb{A}_M)$ such that $r_{l,\imath}(\pi)\cong r$.*

Proof. The proof follows the proof of Proposition 5.2.1 of [BLGG09]. Note that this proof refers only to $r|_{G_M}$ until the very last line where the automorphy of r is deduced from that of $r|_{G_M}$. Also note that in the case that F is imaginary one must replace the argument of the last paragraph of the proof of Proposition 5.2.1 of [BLGG09] with the (similar) argument in the last paragraph of the proof of Proposition 5.1.1 of [BLGG10]. □

Finally recall the following definition from [Tho10]. We will call a subgroup $H\subset GL_n(\overline{\mathbb{F}}_l)$ *adequate* if the following conditions are satisfied.

- $H^1(H,\overline{\mathbb{F}}_l)=(0)$ and $H^1(H,\mathfrak{gl}_n^0(\overline{\mathbb{F}}_l))=(0)$.
- $H^0(H,\mathfrak{gl}_n^0(\overline{\mathbb{F}}_l))=(0)$.
- The elements of H with order coprime to l span $M_{n\times n}(\overline{\mathbb{F}}_l)$ over $\overline{\mathbb{F}}_l$.

Note that this is not exactly the definition given in Definition 2.3 of [Tho10], however it is equivalent to it by Lemma 1 of [GHTT10]. Note also that if H is adequate, then the second condition above implies that $l\not|n$. The following proposition is Theorem 9 of [GHTT10].

Proposition 2.1.2. *Suppose that H is a finite subgroup of $GL_n(\overline{\mathbb{F}}_l)$ such that the tautological representation of H is irreducible. Let H^0 denote the subgroup of H generated by all elements of l-power order and let d denote the maximal dimension of an irreducible H^0-submodule of $\overline{\mathbb{F}}_l^n$. If $l\geq 2(d+1)$ then H is adequate.*

2.2. **Automorphy lifting: the 'minimal' case.**

In this section we present an automorphy lifting theorem which represents the natural output of the Taylor-Wiles-Kisin method (with improvements due to Thorne [Tho10] and Caraiani [Car10]). This result is essentially Theorem 7.1 of [Tho10].

Theorem 2.2.1. *Let F be an imaginary CM field with maximal totally real subfield F^+. Suppose l is an odd prime and that $n \in \mathbb{Z}_{\geq 1}$. Let*

$$r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous representation and let $\overline{r}$ denote the semi-simplification of its reduction. Also let

$$\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$$

be a continuous homomorphism. Suppose that (r, μ) enjoys the following properties:

1. $r^c \cong r^\vee \mu|_{G_F}$.
2. $\mu(c_v) = -1$ *for all* $v|\infty$.
3. *The reduction $\overline{r}$ is (absolutely) irreducible and $\overline{r}(G_{F(\zeta_l)}) \subset GL_n(\overline{\mathbb{F}}_l)$ is adequate.*
4. *There is a RAECSDC automorphic representation (π, χ) of $GL_n(\mathbb{A}_F)$ of level potentially prime to l such that $(\overline{r}, \overline{\mu}) \cong (\overline{r}_{l,\imath}(\pi), \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n})$, and $r_{l,\imath}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}$ for each finite place v of F. (In particular, $r|_{G_{F_v}}$ is potentially crystalline for $v|l$.)*

Then (r, μ) is automorphic of level potentially prime to l.

If further π has level prime to l and if r is crystalline at all primes above l, then (r, μ) is automorphic of level prime to l.

Proof. The result follows from Theorem 7.1 of [Tho10] on noting that for $v \nmid l$ we have $r_{l,\imath}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}$ (by Lemma 1.3.2, because $\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$ and π_v is generic). □

Theorem 2.2.2. *Let F be an imaginary CM field with maximal totally real subfield F^+. Suppose that $n \in \mathbb{Z}_{\geq 1}$ and that l is an odd prime. Let S be a finite set of primes of F^+ including all primes above l. Suppose moreover that each prime in S splits in F and choose a prime $\widetilde{v}$ of F above each $v \in S$. Write $\widetilde{S}$ for the set of $\widetilde{v}$ with $v \in S$.*

Let (π, χ) be a RAECSDC automorphic representation of $GL_n(\mathbb{A}_F)$ which is unramified outside S and has level potentially prime to l. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}\,(F,\mathbb{C}),+}$ be the weight of π. Suppose that the image $\overline{r}_{l,\imath}(\pi)(G_{F(\zeta_l)})$ is adequate.

Suppose that for each $v|l$, $\mathcal{C}_v$ is an irreducible component of

$$\lim_{\to K'} \mathrm{Spec}\, R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}, \{\{a_{\imath\tau,i}+n-i\}_i\}_\tau, K'-\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$$

containing $r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}$. Suppose also that for each $v \in S$ with $v \nmid l$, $\mathcal{C}_v$ is the irreducible component of $\mathrm{Spec}\, R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ containing $r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}$.

Let L denote a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ such that L contains the image of each embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$; and L contains the image of $r_{l,\imath}(\chi)$; and $r_{l,\imath}(\pi)$ is defined over L; and each $\mathcal{C}_v$ is defined over L. For $v \in S$ let $\mathcal{D}_v$ be the deformation problem corresponding to $\mathcal{C}_v$. Also let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}_L, \widetilde{\overline{r}}_{l,\imath}(\pi), r_{l,\imath}(\chi)\epsilon_l^{1-n}, \{\mathcal{D}_v\}_{v\in S}).$$

Then the ring R_S^{univ} is a finitely generated $\mathcal{O}_L$-module.

Proof. Note that $r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}$ lies on a unique component of Spec $R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ for each $v \in S$ with $v \not| l$. (Use Lemma 1.3.2 and the fact that $\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_{\widetilde{v}} \otimes |\det|_{\widetilde{v}}^{(1-n)/2})$.) Also by making a base change to a finite, soluble, Galois, CM extension F'/F which is linearly disjoint from $\overline{F}^{\ker \overline{r}_{l,\imath}(\pi)}(\zeta_l)$ over F we may suppose that π is unramified above l and that $\mathcal{C}_v$ is a component of the spectrum Spec $R^{\square}_{\overline{r}_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}},\{\{a_{\imath\tau,i}+n-i\}_i\}_\tau,\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$ for each $v|l$. (Use Lemma 1.2.2). In particular the character χ is unramified above l (as F/F^+ is unramified above l). The result now follows from Theorem 10.1 of [Tho10]. $\square$

2.3. **Automorphy lifting: the ordinary case.**

One can combine the Taylor-Wiles-Kisin method with the level changing method of [Tay08] and Hida theory, to derive a stronger theorem in the ordinary case. The first such theorem was obtained by D.G. (see Theorem 5.3.2 of [Ger09]). The 'bigness' condition in Theorem 5.3.2 of [Ger09] was relaxed by Thorne. The theorem we present below is Theorem 9.1 of [Tho10].

Theorem 2.3.1. *Let F be an imaginary CM field with maximal totally real subfield F^+. Suppose that l is an odd prime and that $n \in \mathbb{Z}_{\geq 1}$. Let*

$$r : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation and let $\overline{r}$ denote the semi-simplification of the reduction of r. Also let

$$\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^{\times}$$

be a continuous homomorphism. Suppose that (r, μ) enjoys the following properties:

1. *$r^c \cong r^\vee \mu|_{G_F}$.*
2. *$\mu(c_v)$ independent of $v|\infty$.*
3. *r ramifies at only finitely many primes.*
4. *r is ordinary.*
5. *The image $\overline{r}(G_{F(\zeta_l)})$ is adequate and $\zeta_l \notin F$.*
6. *There is a RAECSDC automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
$$(\overline{r}, \overline{\mu}) \cong (\overline{r}_{l,\imath}(\pi), \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n})$$
and π is $\imath$-ordinary.

Then (r, μ) is ordinarily automorphic. If r is also crystalline (resp. potentially crystalline) then (r, μ) is ordinarily automorphic of level prime to l (resp. potentially level prime to l).

The next result is Theorem 10.2 of [Tho10], which generalizes Corollary 4.3.3 of [GG09]. It provides a finiteness theorem for universal deformation rings.

Theorem 2.3.2. *Let F be an imaginary CM field with maximal totally real subfield F^+. Suppose that $n \in \mathbb{Z}_{\geq 1}$ and that l is an odd prime with $\zeta_l \notin F$. Let S be a finite set of primes of F^+ including all primes above l. Suppose moreover that each prime in S splits in F and choose a prime $\widetilde{v}$ of F above each $v \in S$. Write $\widetilde{S}$ for the set of $\widetilde{v}$ for $v \in S$.*

Let (π, χ) be an $\imath$-ordinary RAECSDC automorphic representation of $GL_n(\mathbb{A}_F)$ which is unramified outside S. Suppose that the image $\overline{r}_{l,\imath}(\pi)(G_{F(\zeta_l)})$ is adequate. Let

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

be a continuous de Rham character satisfying $\overline{\mu} = \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n}$. Note that $\mathrm{HT}_\tau(\mu) = \{w\}$ is independent of $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$. For each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ choose a multiset of n distinct integers H_τ such that

$$H_{\tau \circ c} = \{w - h : \ h \in H_\tau\}.$$

Let L denote a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ such that L contains the image of each embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$; and L contains the image of μ; and $r_{l,\imath}(\pi)$ is defined over L. For $v \in S$ with $v \not| l$ let $\mathcal{D}_v$ consist of all lifts of $\overline{r}_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}$. If $v|l$ let $\mathcal{D}_v$ consist of all lifts which factor through $R^\square_{\mathcal{O},\overline{r}_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}},\{H_\tau\},\mathrm{ss\text{-}ord}}$. Also let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}_L, \widetilde{\overline{r}}_{l,\imath}(\pi), \mu, \{\mathcal{D}_v\}_{v \in S}).$$

Then the ring $R^{\mathrm{univ}}_{\mathcal{S}}$ is a finitely generated $\mathcal{O}_L$-module.

3. Potential Automorphy.

3.1. The Dwork family.

In this section we show that a suitable symplectic, mod l representation is potentially automorphic. The theorem and its proof are slight generalizations of section 6 of [BLGHT09]. We start with another minor variant of a result of Moret-Bailly [MB89] (see also [GPR95] and Proposition 6.2 of [BLGHT09]). We apologize for not stating these results in the correct generality the first time round.

Proposition 3.1.1. *Let $D/F/F_0$ be number fields with D/F and F/F_0 Galois. Suppose also that S is a finite set of places of F_0 and let S^F denote the set of places of F above S. For $v \in S^F$ let E'_v/F_v be a finite Galois extension with $E'_{\sigma v} = \sigma E'_v$ for $\sigma \in G_{F_{0,v|_{F_0}}}$. Suppose also that T/F is a smooth, geometrically connected variety and that for each $v \in S^F$ we are given a non-empty, $\mathrm{Gal}\,(E'_v/F_v)$-invariant, open subset $\Omega_v \subset T(E'_v)$.*

Then there is a finite Galois extension E/F and a point $P \in T(E)$ such that

- *E/F_0 is Galois;*
- *E/F is linearly disjoint from D/F;*
- *if $v \in S^F$ and w is a prime of E above v then E_w/F_v is isomorphic to E'_v/F_v and $P \in \Omega_v \subset T(E'_v) \cong T(E_w)$. (This makes sense as Ω_v is $\mathrm{Gal}\,(E'_v/F_v)$-invariant.)*

Proof. Let $D_1, \ldots, D_r$ denote the intermediate fields $D \supset D_i \supset F$ with D_i/F Galois with simple Galois group. Combining Hensel's lemma with the Weil bounds we see that T has an F_v-rational point for all but finitely many primes v of F. Thus enlarging S we may assume that for each $i = 1, \ldots, r$ there is $v \in S^F$ with $E'_v = F_v$ and v not split completely in D_i. Then we may suppress the second condition on E.

Let F'/F be a finite extension such that

- F'/F_0 is Galois;
- if $v \in S^F$ and $w|v$ is a place of F' then F'_w/F_v is isomorphic to E'_v/F_v.

(Apply Lemma 4.1.2 of [CHT08] with F of that lemma our F and S of that lemma our S^F. This produces a soluble extension F''/F. Then we take F' to be the normal closure of F'' over F_0.) Thus we may assume that $E'_v = F_v$ for all $v \in S^F$.

Then Theorem 1.3 of [MB89] tells us that we can find a finite Galois extension F'/F and a point $P \in T(F')$ such that every place v of S^F splits completely in F' and if w is a prime of F' above v then $P \in \Omega_v \subset T(F'_w)$. Now take E to be the normal closure of F' over F_0. □

Theorem 3.1.2. *Suppose that:*

- *F/F_0 is a finite, Galois extension of totally real fields,*
- *$\mathcal{I}$ is a finite set,*
- *for each $i \in \mathcal{I}$, n_i is a positive even integer, l_i is a rational prime, and $\imath_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$,*
- *$F^{(\mathrm{avoid})}/F$ is a finite Galois extension, and*
- *$\bar{r}_i : G_F \to \mathrm{GSp}_{n_i}(\overline{\mathbb{F}}_{l_i})$ is a mod l_i Galois representation with open kernel and multiplier $\bar{\epsilon}_{l_i}^{1-n_i}$.*

Then we can find a finite totally real extension F'/F and RAESDC automorphic representations $(\pi_i', 1)$ of $GL_{n_i}(\mathbb{A}_{F'})$ such that

1. *F'/F_0 is Galois,*
2. *F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F,*
3. *$\overline{r}_{l_i,\imath_i}(\pi_i') \cong \overline{r}_i|_{G_{F'}}$,*
4. *and π_i' has weight 0 and, if $v|l_i$ then $\pi_{i,v}'$ is Steinberg.*

In particular π_i' is $\imath_i$-ordinary and $r_{l_i,\imath_i}(\pi_i')$ is ordinary.

Proof. The proof follows very closely the proof of Theorem 6.3 of [BLGHT09]. We will use the notation of that theorem freely and we will simply indicate the slight changes that need to be made. First note that $\overline{r}_i$ is actually valued in $\mathrm{GSp}_n(\mathbb{F}^{(i)})$ for some finite cardinality subfield $\mathbb{F}^{(i)} \subset \overline{\mathbb{F}}_{l_i}$.

We choose N such that it is not divisible by any rational prime which ramifies in $F^{(\mathrm{avoid})}$ or in any of the $\overline{F}^{\ker \overline{r}_i}(\zeta_{l_i})$, and such that for each $i \in \mathcal{I}$ there is a prime λ_i of $\mathbb{Q}(\zeta_N)^+$ above l_i and an embedding $\mathbb{F}^{(i)} \hookrightarrow \mathbb{Z}[\zeta_N]^+/\lambda_i$. Note that in particular $\mathbb{Q}(\zeta_N)$ is linearly disjoint from $F^{(\mathrm{avoid})}\overline{F}^{\cap_i \ker \overline{r}_i}(\zeta_{\prod_i l_i})$ over $\mathbb{Q}$ and hence also over F.

We choose several fields M_i for $i \in \mathcal{I}$ with $M_i/\mathbb{Q}$ cyclic of degree n_i and with each $M_i/\mathbb{Q}$ unramified at all rational primes ramified in $F^{(\mathrm{avoid})}\overline{F}^{\cap_i \ker \overline{r}_i}(\zeta_{N\prod_i l_i})$. Choose q to be split completely in $\prod_i M_i$ and unramified in $F^{(\mathrm{avoid})}(\zeta_{4N\prod_i l_i})$. Choose $\mathfrak{q}_i$ a prime of M_i above q. Choose M' containing the compositum of the M_i's and characters $\phi_i : \mathbb{A}_{M_i}^\times \to (M')^\times$ as in Theorem 6.3 of [BLGHT09], except that we ask that ϕ_i is unramified above any rational prime which ramifies in $F^{(\mathrm{avoid})}(\zeta_{N\prod_i l_i})$. Choose l' also unramified in $F^{(\mathrm{avoid})}$ and not equal to any l_i and greater than $n_i + 1$ for all i. Choose λ_1 a prime of M' above l' and λ' a prime of $\mathbb{Q}(\zeta_N)^+$ above l'. Use ϕ_i and λ_1 to define $\theta_i : G_{M_i} \to \mathbb{Z}_{l'}^\times$ and set $r_i' = \mathrm{Ind}_{G_{M_i}}^{G_\mathbb{Q}} \theta_i$. Note that $F(\zeta_N)$ and $\overline{\mathbb{Q}}^{\cap_i \ker \overline{r}_i'}(\zeta_{l'})$ are linearly disjoint over $\mathbb{Q}$.

We choose various $T_0^{(i)}$ corresponding to N and n_i. Each comes with a finite cover $T_{\overline{r}_i \times \overline{r}_i'}$. Then choose F' a finite extension of $F(\zeta_N)^+$ and points $t_i \in T_0^{(i)}(F')$ such that

- F'/F_0 is Galois;
- F' is totally real;
- F' is linearly disjoint from $F^{(\mathrm{avoid})}\overline{F}^{\cap_i \ker(\overline{r}_i \times \overline{r}_i')}(\zeta_{l'\prod_i l_i})$ over F;
- $V[\lambda_i]_{t_i} \cong \overline{r}_i|_{G_{F'}}$;
- $V[\lambda']_{t_i} \cong \overline{r}_i'|_{G_{F'}}$;
- $v(t_i) < 0$ for all places $v|l_i$ of F';
- $v(t_i) > 0$ for all places $v|l'$ of F'.

To do this one applies Proposition 3.1.1 with F_0 of that proposition our F_0, with F of that proposition our $F(\zeta_N)^+$ and with $T = \prod_i T_{\overline{r}_i \times \overline{r}_i'}$ and with D of that proposition our $F^{(\mathrm{avoid})}\overline{F}^{\cap_i \ker(\overline{r}_i \times \overline{r}_i')}(\zeta_{l'\prod_i l_i})$. We take S to consist of the places of F_0 dividing infinity or $l'\prod_i l_i$. For $v \in S^{F(\zeta_N)^+}$ and $v|\infty$ we take E_v' to be $F(\zeta_N)_v^+$ and Ω_v to be $T(E_v')$. For $v \in S^{F(\zeta_N)^+}$ with $v|l_i$ (resp. $v|l'$) and E' a finite extension of $F(\zeta_N)_v^+$, let $\Omega(E')$ denote the subset of $T(E')$ consisting of those points whose component in $T_{\overline{r}_i \times \overline{r}_i'}$ lies above an element $t \in T_0^{(i)}(\overline{F}_v)$ with $v(t) < 0$ (resp. with

$v(t) > 0$). For such v we choose finite Galois extensions $E'_v/F(\zeta_N)^+_v$ such that $\Omega(E'_v) \neq \emptyset$ and $E'_{\sigma v} = \sigma E'_v$ for all $\sigma \in G_{F_0, v|_{F_0}}$. We then take $\Omega_v = \Omega(E'_v)$.

The last sentence of the proof of Theorem 6.3 of [BLGHT09] is not required here. $\square$

3.2. **Lifting Galois representations I.**

We now use the method of Khare and Wintenberger [KW09] to show that certain mod l representations have ordinary lifts with prescribed local behaviour. We will later improve upon this by weakening the ordinary hypothesis (see Theorem 4.3.1), but we will need to use this special case before we are in a position to prove the more general result.

Proposition 3.2.1. *Let n be a positive integer and l an odd prime. Suppose that F is a CM field not containing ζ_l and with maximal totally real subfield F^+. Let S be a finite set of finite places of F^+ which split in F and suppose that S includes all places above l. For each $v \in S$ choose a prime $\widetilde{v}$ of F above v.*

Let $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ be a continuous, totally odd, crystalline character unramified outside S. Then there is a $w \in \mathbb{Z}$ such that for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\mu) = \{w\}$. For each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ let H_τ be a set of n distinct integers such that $H_{\tau \circ c} = \{w - h : \ h \in H_\tau\}$.

Let

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

be a continuous representation unramified outside S with $\nu \circ \overline{r} = \overline{\mu}$ and $\overline{r}^{-1}\mathcal{G}_n^0(\overline{\mathbb{F}}_l) = G_F$. Write d for the maximal dimension of an irreducible constituent of the restriction of $\breve{\overline{r}}$ to the closed subgroup of G_{F^+} generated by all Sylow pro-l-subgroups. Suppose that $\breve{\overline{r}}|_{G_{F(\zeta_l)}}$ is irreducible and that $l \geq 2(d+1)$.

Suppose also that for $u|l$ a place of F the restriction $\breve{\overline{r}}|_{G_{F_u}}$ admits a lift $\rho_u : G_{F_u} \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ which is ordinary and crystalline with Hodge-Tate numbers H_τ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$. For $v \in S$ with $v \not| l$ let $\rho_v : G_{F_{\widetilde{v}}} \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ denote a lift of $\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}$.

Then there is lift

$$r : G_{F^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

of $\overline{r}$ such that

1. *$\nu \circ r = \mu$;*
2. *if $u|l$ is a place of F then $\breve{r}|_{G_{F_u}}$ is ordinary and crystalline with Hodge-Tate numbers H_τ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$;*
3. *if $v \in S$ and $v \not| l$ then $\breve{r}|_{G_{F_{\widetilde{v}}}} \sim \rho_v$;*
4. *r is unramified outside S.*

Proof. Choose a place v_q of F above a rational prime q such that v_q is split over F^+ and v_q does not divide any prime in S. Also choose a character $\psi : G_F \to \overline{\mathbb{Q}}_l^\times$ such that

- ψ is unramified above any prime of S which does not divide l;
- ψ is crystalline at all primes above l;
- if $\mathrm{HT}_\tau(\psi) = \{b_\tau\}$ then $|b_\tau - b_{\tau \circ c}| > |h - h'|$ for all $h \in H_\tau, h' \in H_{\tau \circ c}$;
- $q | \#(\psi/\psi^c)(I_{F_{v_q}})$; and
- $\psi\psi^c = \epsilon_l^{1-2n}\mu^{-1}|_{G_F}$.

(See Lemma 4.1.6 of [CHT08].) Consider $I(\overline{r} \otimes (\overline{\psi}, \overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+})) : G_{F^+} \to GSp_{2n}(\overline{\mathbb{F}}_l)$, which has multiplier $\overline{\epsilon}_l^{1-2n}$. By the fourth condition on ψ the representation $I(\overline{r} \otimes (\overline{\psi}, \overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}))|_{G_{F^+(\zeta_l)}}$ is irreducible and so, by Proposition 2.1.2, $I(\overline{r} \otimes (\overline{\psi}, \overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}))(G_{F^+(\zeta_l)})$ is adequate.

Let F_0/F^+ be a totally imaginary quadratic extension linearly disjoint from $\overline{F}^{\ker \mathrm{ad}\, I(\overline{r}\otimes(\overline{\psi},\overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}))}(\zeta_l)$ over F^+. By Theorem 3.1.2 there is a Galois totally real field extension F_1^+/F^+ and a RAESDC automorphic representation $(\pi_1, 1)$ of $GL_{2n}(\mathbb{A}_{F_1^+})$ such that

- F_1^+ is linearly disjoint from $\overline{F}^{\ker I(\overline{r}\otimes(\overline{\psi},\overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}))}F_0(\zeta_l)$ over F^+;
- $\overline{r}_{l,\imath}(\pi_1) \cong I(\overline{r} \otimes (\overline{\psi}, \overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}))|_{G_{F_1^+}}$;
- and π_1 is $\imath$-ordinary.

Set $F_1 = F_0F_1^+$. It is linearly disjoint from $\overline{F}^{\ker \overline{r}\otimes(\overline{\psi},\overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+})}F_1^+(\zeta_l)$ over F_1^+. Set

$$\overline{r}_1 = I((\overline{r} \otimes (\overline{\psi}, \overline{\epsilon}_l^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}))|_{G_{F_1^+}})^{\wedge}_{G_{F_1}} : G_{F_1^+} \longrightarrow \mathcal{G}_{2n}(\overline{\mathbb{F}}_l).$$

Then $\breve{\overline{r}}_1(G_{F_1(\zeta_l)})$ is adequate and $\zeta_l \notin F_1$.

Let $T' \supset S$ denote a finite set of primes of F^+ including all those above which $\overline{\psi}$, π_1 or F_1 is ramified. Let F_2^+/F^+ be a finite soluble Galois totally real extension, linearly disjoint from $\overline{F}_1^{\ker \breve{\overline{r}}_1}(\zeta_l)$ over F^+ such that all primes of $F_3^+ = F_1^+F_2^+$ above T' split in $F_3 = F_1F_2^+$. Set

$$\overline{r}_3 = \overline{r}_1|_{G_{F_3^+}} : G_{F_3^+} \longrightarrow \mathcal{G}_{2n}(\overline{\mathbb{F}}_l)$$

so that $\overline{r}_3^{-1}\mathcal{G}_{2n}^0(\overline{\mathbb{F}}_l) = G_{F_3}$. Then $\breve{\overline{r}}_3(G_{F_3}(\zeta_l))$ is adequate and $\zeta_l \notin F_3$. Let T denote the set of places of F_3^+ lying over T' and for each $u \in T$ choose a prime $\widetilde{u}$ of F_3 above u and let $\widetilde{T}$ denote the set of $\widetilde{u}$ for $u \in T$.

For $v \in S$ with $v \not| l$ let $\mathcal{C}_v$ denote a component of $R^{\square}_{\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ containing ρ_v. Choose a finite extension L of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ with integers $\mathcal{O}$ and residue field $\mathbb{F}$ such that

- L contains the image of each embedding $F_3 \hookrightarrow \overline{\mathbb{Q}}_l$;
- for $v \in S$ the component $\mathcal{C}_v$ is defined over L;
- $\overline{r}$ and $\overline{\psi}$ are defined over $\mathbb{F}$;
- and μ is defined over L.

For $v \in S$ with $v \not| l$ let $\mathcal{D}_v$ denote the deformation problem for $\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}$ corresponding to $\mathcal{C}_v$. For $v \in S$ with $v|l$ let $\mathcal{D}_v$ consist of all lifts of $\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}$ which factor through $R^{\square}_{\mathcal{O},\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}},\{H_\tau\},\mathrm{cr\text{-}ord}}$. Set

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \mu, \{\mathcal{D}_v\}_{v\in S}).$$

For $u \in T$ with $u \not| l$ let $\mathcal{D}_{3,u}$ consist of all lifts of $\breve{\overline{r}}_3|_{G_{F_{3,\widetilde{u}}}}$. For $u \in T$ with $u|l$ let $\mathcal{D}_{3,u}$ consist of all lifts of $\breve{\overline{r}}_3|_{G_{F_{3,\widetilde{u}}}}$ which factor through $R^{\square}_{\mathcal{O},\breve{\overline{r}}_3|_{G_{F_{3,\widetilde{u}}}},\{H_{3,\tau}\},\mathrm{ss\text{-}ord}}$, where

$$H_{3,\tau} = \{h + b_{\tau_1} : \ h \in H_{\tau_1}\} \cup \{h + b_{\tau_2} : \ h \in H_{\tau_2}\},$$

and τ_1 and τ_2 denote the two embeddings of $F \hookrightarrow \overline{\mathbb{Q}}_l$ lying above $\tau|_{F^+}$. Set

$$\mathcal{S}_3 = (F_3/F_3^+, T, \widetilde{T}, \mathcal{O}, \overline{r}_3, \epsilon_l^{1-2n}, \{\mathcal{D}_{3,u}\}_{u \in T}).$$

According to Theorem 2.3.2 the ring $R_{\mathcal{S}_3}^{\mathrm{univ}}$ is a finitely generated $\mathcal{O}$-module. Hence by Lemma 1.2.2 the ring $R_{\mathcal{S}}^{\mathrm{univ}}$ is also a finitely generated $\mathcal{O}$-module. On the other hand by Proposition 1.5.1 $R_{\mathcal{S}}^{\mathrm{univ}}$ has Krull dimension at least 1 and so there is a continuous ring homomorphism $R_{\mathcal{S}}^{\mathrm{univ}} \to \overline{\mathbb{Q}}_l$. The push forward of the universal deformation of $\overline{r}$ by this homomorphism is our desired lift r. $\square$

3.3. **Potential ordinary automorphy.**

In this section we improve Theorem 3.1.2 to show that a suitable mod l representation is potentially ordinarilly automorphic with prescribed "weight and level". The proof will combine Theorem 3.1.2 and Proposition 3.2.1. We will improve further on this result in Corollary 4.5.3.

Proposition 3.3.1. *Suppose that we are in the following situation.*

(a) *Let F/F_0 be a finite, Galois extension of CM fields, and let F^+ and F_0^+ denote their maximal totally real subfields. Choose a complex conjugation $c \in G_{F^+}$.*
(b) *Let $\mathcal{I}$ be a finite set.*
(c) *For each $i \in \mathcal{I}$ let n_i be a positive integer and l_i be an odd rational prime with $\zeta_{l_i} \notin F$. Also choose $\imath_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$ for each $i \in \mathcal{I}$.*
(d) *For each $i \in \mathcal{I}$ let $\mu_i : G_{F^+} \to \overline{\mathbb{Q}}_{l_i}^{\times}$ be a continuous, totally odd, de Rham character. Then there is a $w_i \in \mathbb{Z}$ such that for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ we have $\mathrm{HT}_\tau(\mu_i) = \{w_i\}$.*
(e) *For each $i \in \mathcal{I}$ let $\overline{r}_i : G_F \to GL_{n_i}(\overline{\mathbb{F}}_{l_i})$ be an irreducible continuous representation such that $(\overline{r}_i, \overline{\mu}_i)$ is totally odd, essentially conjugate self-dual. Let d_i denote the maximal dimension of an irreducible sub-representation of the restriction of $\overline{r}_i$ to the subgroup of G_F generated by all Sylow pro-l_i-subgroups. Suppose that $\overline{r}_i|_{G_{F(\zeta_{l_i})}}$ is irreducible and that $l_i \geq 2(d_i + 1)$.*
(f) *For each $i \in \mathcal{I}$ and each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ let $H_{i,\tau}$ be a set of n_i distinct integers such that $H_{i,\tau \circ c} = \{w_i - h : \ h \in H_{i,\tau}\}$.*
(g) *Let S denote a finite $\mathrm{Gal}\,(F/F^+)$-invariant set of primes of F including all those dividing $\prod_i l_i$ and all those at which some $\overline{r}_i$ ramifies.*
(h) *for each $i \in \mathcal{I}$ and $v \in S$ with $v \not| l_i$ let $\rho_{i,v} : G_{F_v} \to GL_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ denote a lift of $\overline{r}_i|_{G_{F_v}}$ such that $\rho_{i,cv}^c \cong \mu_i|_{G_{F_v}} \rho_{i,v}^\vee$.*
(i) *Let $F^{(\mathrm{avoid})}/F$ be a finite Galois extension.*

Then we can find a finite CM extension F'/F and for each $i \in \mathcal{I}$ a RAECSDC automorphic representation (π_i, χ_i) of $GL_{n_i}(\mathbb{A}_{F'})$ such that

(1) *F'/F_0 is Galois,*
(2) *F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F,*
(3) *$\overline{r}_{l_i,\imath_i}(\pi_i) \cong \overline{r}_i|_{G_{F'}}$;*
(4) *$r_{l_i,\imath_i}(\chi_i)\epsilon_{l_i}^{1-n_i} = \mu_i|_{G_{(F')^+}}$;*
(5) *π_i is unramified above l_i and outside S;*
(6) *π_i is $\imath_i$-ordinary;*
(7) *if $\tau : F' \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ then $\mathrm{HT}_\tau(r_{l_i,\imath_i}(\pi_i)) = H_{i,\tau|_F}$;*

(8) *if $u \nmid l_i$ is a prime of F' lying above an element $v \in S$ then* $r_{l_i,\imath_i}(\pi_i)|_{G_{F'_u}} \sim \rho_{i,v}|_{G_{F'_u}}$.

Proof. Note that $(\overline{r}_i, \overline{\mu}_i|_{G_F})$ extends to a continuous homomorphism $\tilde{\overline{r}}_i : G_{F^+} \to \mathcal{G}_{n_i}(\overline{\mathbb{F}}_{l_i})$ with $\nu \circ \tilde{\overline{r}}_i = \overline{\mu}_i$ (see section 1.1).

Choose a finite totally real extension F_1^+/F^+ so that

- F_1^+/F_0^+ is Galois;
- F_1^+ is linearly disjoint from $\overline{F}^{\bigcap_i \ker \overline{r}_i}(\zeta_{\prod_i l_i})F^{(\mathrm{avoid})}$ over F^+;
- all places of $F_1 = FF_1^+$ above S are split over F_1^+;
- and for all $i \in \mathcal{I}$ and all places $u|l_i$ of F_1 the restriction $\overline{r}_i|_{G_{F_{1,u}}}$ admits a lift $\rho_{i,u} : G_{F_{1,u}} \to GL_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ which is ordinary and crystalline with Hodge-Tate numbers $H_{i,\tau|_F}$ for each $\tau : F_{1,u} \hookrightarrow \overline{\mathbb{Q}}_l$.

(This is possible by Proposition 3.1.1 applied with $T = \operatorname{Spec} F^+$.) Replacing F by F_1 (and $F^{(\mathrm{avoid})}$ by $F_1F^{(\mathrm{avoid})}$) we may reduce the theorem to the special case that all elements of S are split over F^+ and that for all $i \in \mathcal{I}$ and all places $u|l_i$ of F the restriction $\overline{r}_i|_{G_{F_u}}$ admits a lift $\rho_{i,u} : G_{F_u} \to GL_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ which is ordinary and crystalline with Hodge-Tate numbers $H_{i,\tau}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$. (Note that if F'/F_1 is finite and linearly disjoint from $F^{(\mathrm{avoid})}F_1$ over F_1 and if F'/F_0 is Galois, then F'/F is linearly disjoint from $F^{(\mathrm{avoid})}$ over F.)

In this case, using Proposition 3.2.1, we see that for each $i \in \mathcal{I}$ there is a lift

$$r_i : G_{F^+} \longrightarrow \mathcal{G}_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$$

of $\tilde{\overline{r}}_i$ such that

- $\nu \circ r_i = \mu_i$;
- if $u|l_i$ is a place of F then $\breve{r}_i|_{G_{F_u}}$ is ordinary and crystalline with Hodge-Tate numbers $H_{i,\tau}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_{l_i}$;
- if $v \in S$ and $v \nmid l_i$ then $\breve{r}_i|_{G_{F_v}} \sim \rho_{i,v}$;
- r_i is unramified outside S.

(Note that if we write $S = \widetilde{S} \coprod c\widetilde{S}$ then we only need check the penultimate assertion for $v \in \widetilde{S}$ and it will follow also for $v \in c\widetilde{S}$.)

Choose a place v_q of F which is split over F^+ and which lies above a rational prime q, which in turn does not lie under any prime in S. Also choose characters $\psi_i : G_F \to \overline{\mathbb{Q}}_{l_i}^\times$ for $i \in \mathcal{I}$ such that

- ψ_i is unramified at places in S which do not divide l_i;
- ψ_i is crystalline at all places above l_i;
- if $\mathrm{HT}_\tau(\psi_i) = \{b_{i,\tau}\}$ then $|b_{i,\tau} - b_{i,\tau\circ c}| > |h - h'|$ for all $h \in H_\tau$ and $h' \in H_{\tau \circ c}$;
- $q|\#(\psi_i/\psi_i^c)(I_{F_{v_q}})$; and
- $\psi_i\psi_i^c = \epsilon_{l_i}^{1-2n_i}\mu_i^{-1}|_{G_F}$.

(See Lemma 4.1.6 of [CHT08].) Consider $I(\tilde{\overline{r}}_i \otimes (\overline{\psi}_i, \overline{\epsilon}_{l_i}^{1-2n_i}\overline{\mu}_i^{-1}\delta_{F/F^+})) : G_{F^+} \to GSp_{2n_i}(\overline{\mathbb{F}}_{l_i})$, which has multiplier $\overline{\epsilon}_{l_i}^{1-2n_i}$. As in the proof of Proposition 3.2.1 we see that $I(\tilde{\overline{r}}_i \otimes (\overline{\psi}_i, \overline{\epsilon}_{l_i}^{1-2n_i}\overline{\mu}_i^{-1}\delta_{F/F^+}))(G_{F^+(\zeta_{l_i})})$ is adequate.

Theorem 3.1.2 tells us that there is a finite totally real field extension F_1^+/F^+ and RAESDC automorphic representations $(\pi_{1,i}, 1)$ of $GL_{2n_i}(\mathbb{A}_{F_1^+})$ such that

- F_1^+/F_0^+ is Galois;

- F_1^+ is linearly disjoint from $\overline{F}^{\cap_i \ker I(\tilde{\overline{r}}_i \otimes (\overline{\psi}_i, \overline{\epsilon}_{l_i}^{1-2n_i} \overline{\mu}_i^{-1} \delta_{F/F^+}))}(\zeta_{\prod_i l_i})F^{(\mathrm{avoid})}$ over F^+;
- $\overline{r}_{l_i,\imath_i}(\pi_{1,i}) \cong I(\tilde{\overline{r}}_i \otimes (\overline{\psi}_i, \overline{\epsilon}_{l_i}^{1-2n_i} \overline{\mu}_i^{-1} \delta_{F/F^+}))|_{G_{F_1^+}}$
- $\pi_{1,i}$ has weight 0;
- and if $v|l_i$ is a prime of F_1^+, then $\pi_{1,i,v}$ is Steinberg.

Let F_1 be a totally imaginary quadratic extension of F_1^+ which is linearly disjoint from $\overline{F}^{\cap_i \ker I(\tilde{\overline{r}}_i \otimes (\overline{\psi}_i, \overline{\epsilon}_{l_i}^{1-2n_i} \overline{\mu}_i^{-1} \delta_{F/F^+}))|_{G_{F_1^+}}}(\zeta_{\prod_i l_i})F^{(\mathrm{avoid})}$ over F_1^+. Note that π_{1,i,F_1} has weight 0 and is Steinberg above l_i, and hence is $\imath_i$-ordinary. By Theorem 2.3.1 we conclude that for each $i \in \mathcal{I}$ there is an $\imath_i$-ordinary RAECSDC automorphic representation $(\pi_{2,i}, 1)$ of $GL_{2n_i}(\mathbb{A}_{F_1})$ of level prime to l_i such that $r_{l_i,\imath_i}(\pi_{2,i}) = I(\tilde{r}_i \otimes (\psi_i, \epsilon_{l_i}^{1-2n_i} \mu_i^{-1} \delta_{F/F^+}))|_{G_{F_1}}$. Hence (by Lemma 5.1.6 of [Ger09] and Lemma 1.5 of [BLGHT09]) there is an $\imath_i$-ordinary RAESDC automorphic representation $(\pi_{3,i}, 1)$ of $GL_{2n_i}(\mathbb{A}_{F_1^+})$ of level prime to l_i such that $r_{l_i,\imath_i}(\pi_{3,i}) = I(\tilde{r}_i \otimes (\psi_i, \epsilon_{l_i}^{1-2n_i} \mu_i^{-1} \delta_{F/F^+}))|_{G_{F_1^+}}$.

Let $F' = FF_1^+$. By Lemma 2.1.1, we see that for each $i \in \mathcal{I}$ there is a RAECSDC representation (π'_i, χ'_i) of $GL_{n_i}(\mathbb{A}_{F'})$ of level prime to l_i with $r_{l_i,\imath_i}(\pi'_i) = (\breve{r}_i \otimes \psi_i)|_{G_{F'}}$. The theorem follows (using local-global compatibility).

$\square$

4. The main theorems.

4.1. A preliminary automorphy lifting result.

The proof of the next proposition is our main innovation. The last two parts of assumption (5) are rather restrictive and mean that the proposition is not directly terribly useful. However in the next section we will see how we can combine this result with Theorem 2.3.1 to get a genuinely useful result. Our main tool will be Harris' tensor product trick (see [Har09] and [BLGHT09]).

Proposition 4.1.1. *Let F be an imaginary CM field with maximal totally real subfield F^+ and let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. Suppose that l is an odd prime and let $n \in \mathbb{Z}_{\geq 1}$. Let*

$$r : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation and let $\overline{r}$ denote the semi-simplification of the reduction of r. Let d denote the maximal dimension of an irreducible subrepresentation of the restriction of $\overline{r}$ to the closed subgroup of G_F generated by all Sylow pro-l-subgroups. Also let

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$

be a continuous character. Suppose that r and μ enjoy the following properties:

1. *$r^c \cong r^\vee \mu$.*
2. *r ramifies at only finitely many primes.*
3. *$r|_{G_{F_v}}$ is potentially diagonalizable (and so in particular potentially crystalline) for all $v|l$, and for each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ it has n distinct τ-Hodge-Tate numbers.*
4. *The restriction $\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible and $l \geq 2(d+1)$.*
5. *There is a RAECSDC automorphic representation (π, χ) of $GL_n(\mathbb{A}_F)$ such that*
$$(\overline{r}, \overline{\mu}) \cong (\overline{r}_{l,\imath}(\pi), \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n})$$
and
    - *π has level potentially prime to l;*
    - *$r_{l,\imath}(\pi)|_{G_{F_v}}$ is potentially diagonalizable for all $v|l$;*
    - *for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ the set $\{h + h' : \ h \in \mathrm{HT}_\tau(r),\ h' \in \mathrm{HT}_\tau(r_{l,\imath}(\pi))\}$ has n^2 distinct elements;*
    - *if $v \not| l$ then $r_{l,\imath}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}$.*

Then (r, μ) is automorphic of level potentially prime to l.

Proof. Note that $\imath\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{F-\mathrm{ss}} = \mathrm{rec}(\pi_v | \ |_v^{(1-n)/2})$ for all $v \not| l$. Moreover as π_v is generic we have $r_{l,\imath}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}$ for all $v \not| l$.

Using Lemma 1.4 of [BLGHT09] (base change) it is enough to prove the theorem after replacing F by a soluble CM extension which is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F. Thus we may suppose that

- F/F^+ is unramified at all finite primes;
- all primes dividing l and all primes at which π or r ramify are split over F^+;
- if u is a place of F above a rational prime which equals l or above which π ramifies, then $\overline{r}|_{G_{F_u}}$ is trivial;

- if u is a place of F above l then $r|_{G_{F_u}}$ and $r_{l,\imath}(\pi)|_{G_{F_u}}$ are diagonalizable, and π_u is unramified.

For each prime v of F^+ which splits in F, choose once and for all a prime $\widetilde{v}$ of F above v.

For u a prime of F above l suppose that

$$r|_{G_{F_u}} \sim \psi_1^{(u)} \oplus \cdots \oplus \psi_n^{(u)},$$

and

$$r_{l,\imath}(\pi)|_{G_{F_u}} \sim \phi_1^{(u)} \oplus \cdots \oplus \phi_n^{(u)},$$

for crystalline characters $\psi_i^{(u)}$ and $\phi_i^{(u)} : G_{F_u} \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$. We can, and shall, assume that the characters $\psi_i^{(u)}$ and $\phi_i^{(u)}$ satisfy $\psi_i^{(cu)}(\psi_i^{(u)})^c = \mu|_{G_{F_{cu}}}$ and $\phi_i^{(cu)}(\phi_i^{(u)})^c = (r_{l,\imath}(\chi)\epsilon_l^{1-n})|_{G_{F_{cu}}}$. For $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$ write $\mathrm{HT}_\tau(\psi_i^{(u)}) = \{h'_{\tau,i}\}$ and $\mathrm{HT}_\tau(\phi_i^{(u)}) = \{h_{\tau,i}\}$. There are integers w and w' such that for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\mu) = \{w'\}$ and $\mathrm{HT}_\tau(r_{l,\imath}(\chi)) = \{w+1-n\}$. Then

$$h'_{\tau,i} + h'_{\tau c,i} = w'$$

and

$$h_{\tau,i} + h_{\tau c,i} = w$$

for all τ and i.

Choose a CM extension M/F such that

- M/F is cyclic of degree n;
- M is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F;
- and all primes of F above l or at which π ramifies split completely in M.

Choose a prime u_q of F above a rational prime q such that

- $q \neq l$ and q splits completely in M;
- r, μ, π and χ are unramified above q.

If $v|ql$ is a prime of F we label the primes of M above v as $v_{M,1}, \ldots, v_{M,n}$ so that $(cv)_{M,i} = c(v_{M,i})$. Choose continuous characters

$$\theta, \theta' : G_M \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that

- the reductions $\overline{\theta}$ and $\overline{\theta}'$ are equal;
- $\theta\theta^c = r_{l,\imath}(\chi)\epsilon_l^{1-n}$ and $\theta'(\theta')^c = \mu$;
- θ and θ' are de Rham;
- if $\tau : M \hookrightarrow \overline{\mathbb{Q}}_l$ lies above a place $v_{M,i}|l$ of M then $\mathrm{HT}_\tau(\theta) = \{h_{\tau|_F,i}\}$ and $\mathrm{HT}_\tau(\theta') = \{h'_{\tau|_F,i}\}$;
- θ and θ' are unramified at $u_{q,M,i}$ for $i > 1$, but q divides $\#\theta(I_{M_{u_{q,M,1}}})$ and $\#\theta'(I_{M_{u_{q,M,1}}})$.

(Use Lemma 4.1.6 of [CHT08] twice.) Note that if $u|l$ is a place of F and if K/F_u is a finite extension over which θ and θ' become crystalline and $\overline{\theta} = \overline{\theta}'$ become trivial, then

$$(\mathrm{Ind}_{G_M}^{G_F} \theta)|_{G_K} \sim \phi_1^{(u)}|_{G_K} \oplus \cdots \oplus \phi_n^{(u)}|_{G_K}$$

and

$$(\mathrm{Ind}_{G_M}^{G_F} \theta')|_{G_K} \sim \psi_1^{(u)}|_{G_K} \oplus \cdots \oplus \psi_n^{(u)}|_{G_K}.$$

Now let F_1/F be a solvable CM extension such that

- $\theta|_{G_{F_1M}}$ and $\theta'|_{G_{F_1M}}$ are unramified away from l and crystalline at all primes above l;
- $\bar{\theta}|_{G_{F_1M}}$ is trivial at all primes above l;
- F_1 is linearly disjoint from $\overline{F}^{\ker(\overline{r}\otimes\operatorname{Ind}_{G_M}^{G_F}\overline{\theta})}M(\zeta_l)$ over F;
- MF_1/F_1 is unramified at all finite places.

Note that MF_1/F_1 is split completely above all places of F at which π is ramified.

Put

$$R := (r \otimes (\operatorname{Ind}_{G_M}^{G_F}\theta))|_{G_{F_1}},$$
$$R' := (r_{l,\imath}(\pi) \otimes (\operatorname{Ind}_{G_M}^{G_F}\theta'))|_{G_{F_1}}.$$

Note that we have the following facts.

- $\overline{R} \cong \overline{R}'$.
- $R^c \cong (r^\vee \otimes \operatorname{Ind}_{G_M}^{G_F}\theta^c \otimes \mu)|_{G_{F_1}} \cong R^\vee \otimes (\mu r_{l,\imath}(\chi)\epsilon_l^{1-n})|_{G_{F_1}}$.
- $(R')^c \cong (R')^\vee(\mu r_{l,\imath}(\chi)\epsilon_l^{1-n})|_{G_{F_1}}$.
- $\overline{R}$ is irreducible and $\overline{R}|_{G_{F_1(\zeta_l)}}$ is adequate.

  [As $\overline{r}|_{G_{M(\zeta_l)}}$ is irreducible, we see that the restriction to $G_{M(\zeta_l)}$ of any constituent of $(\overline{r}\otimes\operatorname{Ind}_{G_M}^{G_F}\overline{\theta})|_{G_{F(\zeta_l)}}$ is a sum of $\overline{r}|_{G_{M(\zeta_l)}}\overline{\theta}^\tau|_{G_{M(\zeta_l)}}$ as τ runs over some subset of $\operatorname{Gal}(M/F)$. Looking at ramification above u_q we see that the $\overline{r}|_{G_{M(\zeta_l)}}\overline{\theta}^\tau|_{G_{M(\zeta_l)}}$ are permuted transitively by $\operatorname{Gal}(M/F)$ and hence $(\overline{r}\otimes\operatorname{Ind}_{G_M}^{G_F}\overline{\theta})|_{G_{F(\zeta_l)}}$ is irreducible. By the choice of F_1, we see that $\overline{R}|_{G_{F_1(\zeta_l)}}$ is irreducible. By the argument of the second paragraph of the proof of Theorem 9 of [GHTT10] we see that $l\not| n$ and hence every Sylow pro-l subgroup of $G_{F(\zeta_l)}$ is a subgroup of $G_{M(\zeta_l)}$. By Proposition 2.1.2, we see that $\overline{R}|_{G_{F_1(\zeta_l)}}$ is adequate.]
- $(R', \mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+})$ is automorphic of level prime to l, say

  $$(R', \mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+}) \cong (r_{l,\imath}(\pi_1), r_{l,\imath}(\chi_1)\epsilon_l^{1-n^2}).$$

  Moreover π_1 only ramifies at places of F_1 where π_{F_1} is ramified. (π_1 is constructed as the automorphic induction of $\pi_{F_1M} \otimes (\phi'| \;|^{n(n-1)/2} \circ \det)$ to F_1, where $r_{l,\imath}(\phi') = \theta'|_{G_{F_1M}}$. Note that if $\sigma \in \operatorname{Gal}(F_1M/F_1)$ then $\overline{r}_{l,\imath}(\pi)|_{G_{F_1M}}\overline{\theta}'|^\sigma_{G_{F_1M}} \not\cong \overline{r}_{l,\imath}(\pi)|_{G_{F_1M}}\overline{\theta}'|_{G_{F_1M}}$, so that $(\pi_{F_1M} \otimes (\phi' \circ \det))^\sigma \not\cong \pi_{F_1M} \otimes (\phi' \circ \det)$ and π_1 is cuspidal.)
- For all places $u|l$ of F_1,

$$\begin{aligned} R|_{G_{F_{1,u}}} &\sim (\psi_1^{(u|_F)} \oplus \cdots \oplus \psi_n^{(u|_F)})|_{G_{F_{1,u}}} \otimes (\phi_1^{(u|_F)} \oplus \cdots \oplus \phi_n^{(u|_F)})|_{G_{F_{1,u}}} \\ &\sim R'|_{G_{F_{1,u}}}. \end{aligned}$$

- For all places $u \nmid l$ of F_1 we have $R'|_{G_{F_{1,u}}} \sim R_{G_{F_{1,u}}}$. [Because we know that $r_{l,\imath}(\pi_{F_1})|_{G_{F_{1,u}}} \sim r|_{G_{F_{1,u}}}$ and $(\operatorname{Ind}_{G_M}^{G_F}\theta')|_{G_{F_{1,u}}} \sim (\operatorname{Ind}_{G_M}^{G_F}\theta)|_{G_{F_{1,u}}}$, the latter because they are both unramified.]

We now apply Theorem 2.2.1, with

- F the present F_1,
- l as in the present setting,
- n the present n^2,
- r the present R,
- μ the present $\mu r_{l,\imath}(\chi)\epsilon_l^{1-n}\delta_{F_1/F_1^+}$,

- (π, χ) the present (π_1, χ_1).

We conclude that R is automorphic of level prime to l. By Lemma 2.1.1, $r|_{G_{F_1}}$ is automorphic of level prime to l. Finally by Lemma 1.4 of [BLGHT09], r is automorphic (of level prime to l). $\square$

4.2. **Automorphy lifting: the potentially diagonal case.**

In this section we will prove our main automorphy lifting theorem. It generalizes Theorem 2.3.1 from the ordinary case to the potentially diagonalizable case. It is proved by combining Theorem 2.3.1 and Propositions 3.2.1 and 4.1.1.

Theorem 4.2.1. *Let F be an imaginary CM field with maximal totally real subfield F^+ and let c denote the non-trivial element of $\mathrm{Gal}\,(F/F^+)$. Suppose that $n \in \mathbb{Z}_{\geq 1}$ and that l is an odd prime. Let*

$$r : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

be a continuous irreducible representation and let $\overline{r}$ denote the semi-simplification of the reduction of r. Let d denote the maximal dimension of an irreducible sub-representation of the restriction of $\overline{r}$ to the closed subgroup of G_F generated by all Sylow pro-l-subgroups. Also let

$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

be a continuous character. Suppose that r and μ enjoy the following properties:

1. *$r^c \cong r^{\vee}\mu$.*
2. *r ramifies at only finitely many primes.*
3. *$r|_{G_{F_v}}$ is potentially diagonalizable (and so in particular potentially crystalline) for all $v|l$ and for each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ it has n distinct τ-Hodge-Tate numbers.*
4. *The restriction $\overline{r}|_{G_{F(\zeta_l)}}$ is irreducible, $l \geq 2(d+1)$, and $\zeta_l \notin F$.*
5. *There is a RAECSDC automorphic representation (π, χ) of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
$$(\overline{r}, \overline{\mu}) \cong (\overline{r}_{l,\imath}(\pi), \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n}).$$
   Suppose further that
   - *either π is $\imath$-ordinary,*
   - *or π has level potentially prime to l and $r_{l,\imath}(\pi)|_{G_{F_v}}$ is potentially diagonalizable for all $v|l$.*

Then (r, μ) is automorphic of level potentially prime to l.

We remark that condition 3 of the theorem will be satisfied if, in particular, l is unramified in F and r is crystalline at all primes above l, and $\mathrm{HT}_\tau(r)$ is contained in an interval of the form $[a_\tau, a_\tau + l - 2]$ for all τ (the "Fontaine-Laffaille" case). We also remark that the reason we can not immediately apply Proposition 4.1.1 to deduce this theorem is the last two parts of assumption 5 in Proposition 4.1.1 (i.e. roughly speaking r and $r_{l,\imath}(\pi)$ may have different level or $r \otimes r_{l,\imath}(\pi)$ may have repeated Hodge-Tate weights). To get round this problem we use Proposition 3.2.1 to create two ordinary intermediate lifts of $\overline{r}$, one r_1 with similar behaviour ('level') to r and one r_2 with similar behaviour to $r_{l,\imath}(\pi)$. We also ensure that $r_1 \otimes r$ and $r_2 \otimes r_{l,\imath}(\pi)$ are Hodge-Tate regular. Theorem 2.3.1 tells us that if r_2 is automorphic so is r_1. On the other hand Proposition 4.1.1 allows one to show that r_2 is automorphic and that if r_1 is automorphic then so is r.

Proof. Using Lemma 1.4 of [BLGHT09] (base change) it is enough to prove the theorem after replacing F by a soluble CM extension which is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F. Thus we may suppose that

- F/F^+ is unramified at all finite primes;
- all primes dividing l and all primes at which π or r ramify are split over F^+;
- if u is a place of F above l then F_u contains a primitive l^{th} root of unity, and $\overline{r}|_{G_{F_u}}$ and $\overline{r}_{l,\imath}(\pi)|_{G_{F_u}}$ are trivial.

Let S denote the set of primes of F^+ which divide l or above which r or π ramifies. For each prime $v \in S$ choose once and for all a prime $\widetilde{v}$ of F above v.

Note that $\mu(c) = (-1)$ for all complex conjugations c and that we may extend $\overline{r} \cong \overline{r}_{l,\imath}(\pi)$ to a homomorphism

$$\tilde{\overline{r}} : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

with multiplier $\overline{\mu}$.

Choose an integer m strictly greater than $|h - h'|$ for all h and h', Hodge-Tate numbers for r or $r_{l,\imath}(\pi)$. If $\tau : F \hookrightarrow \mathbb{C}$ set

$$H_\tau = \{0, m, \ldots, (n-1)m\}.$$

Note that if $u|l$ then both $\overline{r}|_{G_{F_u}}$ and $\overline{r}_{l,\imath}(\pi)|_{G_{F_u}}$ have ordinary and crystalline lifts $1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}$ with τ-Hodge-Tate numbers $H_{\tau|_F}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$. Applying Proposition 3.2.1 we see that there is a continuous homomorphism

$$r_1 : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

lifting $\tilde{\overline{r}}$ and such that

- $\nu \circ r_1 = \epsilon_l^{(1-n)m} \omega_l^{(n-1)m} \widetilde{\mu}$ where $\widetilde{\mu}$ denotes the Teichmuller lift of $\overline{\mu}$;
- if $u|l$ then $\breve{r}_1|_{G_{F_u}}$ is ordinary and crystalline with Hodge-Tate numbers $H_{\tau|_F}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$;
- r_1 is unramified outside S;
- and if $v \in S$ and $v \not| l$ then $r|_{G_{F_{\widetilde{v}}}} \sim \breve{r}_1|_{G_{F_{\widetilde{v}}}}$.

Suppose we are in the first case of assumption (5). Then Theorem 2.3.1 tells us that $\breve{r}_1$ is automorphic of level prime to l. Then Proposition 4.1.1 tells us that r is automorphic of level potentially prime to l.

Suppose now that we are in the second case of assumption (5). Again applying Proposition 3.2.1 we find a continuous homomorphism

$$r_2 : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

lifting $\tilde{\overline{r}}$ and such that

- $\nu \circ r_2 = \epsilon_l^{(1-n)m} \omega_l^{(n-1)(m-1)} \widetilde{\chi}$ where $\widetilde{\chi}$ denotes the Teichmuller lift of $\overline{r}_{l,\imath}(\chi)$;
- if $u|l$ then $\breve{r}_2|_{G_{F_u}}$ is ordinary and crystalline with Hodge-Tate numbers $H_{\tau|_F}$ for each $\tau : F_u \hookrightarrow \overline{\mathbb{Q}}_l$;
- r_2 is unramified outside S;
- and if $v \in S$ and $v \not| l$ then $r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}} \sim \breve{r}_2|_{G_{F_{\widetilde{v}}}}$.

Suppose that $r_2 = r_{l,\imath}(\pi_2)$. As r_2 is ordinary and π_2 has level potentially prime to l we can conclude that π_2 is $\imath$-ordinary, and we are reduced to the first case of assumption (5). $\square$

4.3. **Lifting Galois representations II.**

We now use the same idea that we used to prove Theorem 4.2.1 to prove a strengthening of Proposition 3.2.1.

Theorem 4.3.1. *Let n be a positive integer and l an odd prime. Suppose that F is a CM field not containing ζ_l and with maximal totally real subfield F^+. Let S be a finite set of finite places of F^+ which split in F and suppose that S includes all places above l. For each $v \in S$ choose a prime $\widetilde{v}$ of F above v.*

Let $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ be a continuous, totally odd, de Rham character unramified outside S. Also let

$$\overline{r} : G_{F^+} \longrightarrow \mathcal{G}_n(\overline{\mathbb{F}}_l)$$

be a continuous representation unramified outside S with $\nu \circ \overline{r} = \overline{\mu}$ and $\overline{r}^{-1}\mathcal{G}_n^0(\overline{\mathbb{F}}_l) = G_F$. Suppose that $\breve{\overline{r}}|_{G_{F(\zeta_l)}}$ is irreducible.

Let d denote the maximal dimension of an irreducible sub-representation of the restriction of $\overline{r}$ to the closed subgroup of G_F generated by all Sylow pro-l-subgroups. Suppose that $l \geq 2(d+1)$.

For $v \in S$, let $\rho_v : G_{F_{\widetilde{v}}} \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ denote a lift of $\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}$. If $v|l$ we assume that ρ_v is potentially diagonalizable and that, for all $\tau : F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$, the multiset $\mathrm{HT}_\tau(\rho_v)$ consists of n distinct integers.

Then there is a lift

$$r : G_{F^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

of $\overline{r}$ such that

1. *$\nu \circ r = \mu$;*
2. *if $v \in S$ then $\breve{r}|_{G_{F_{\widetilde{v}}}} \sim \rho_v$;*
3. *r is unramified outside S.*

Proof. We may suppose that for $v \in S$ with $v \not| l$ the representation ρ_v is robustly smooth (see Lemma 1.3.2) and hence lies on a unique component $\mathcal{C}_v$ of $R^{\square}_{\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$. If $v|l$ is a place of F^+ then choose a finite extension $K_v/F_{\widetilde{v}}$ over which ρ_v becomes crystalline, and let $\mathcal{C}_v$ denote the unique component of $R^{\square}_{\breve{\overline{r}}|_{G_{F_{\widetilde{v}}}},\{\mathrm{HT}_\tau(\rho_v)\},K_v-\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$ on which ρ_v lies. Let $\widetilde{\mu}$ denote the Teichmuller lift of $\overline{\mu}$. Choose a positive integer m which is greater than one plus the difference of every two Hodge-Tate numbers of ρ_v for every $v|l$.

Choose a finite, soluble, Galois, CM extension F_1/F which is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F such that

- for all u lying above S we have $\overline{r}(G_{F_{1,u}}) = \{1\}$;
- for all $u|l$ we have $\zeta_l \in F_{1,u}$;
- $\mu|_{G_{F_1^+}}$ is crystalline above l;
- if $u|\widetilde{v}|l$ with $v \in S$ then $\rho_v|_{G_{F_{1,u}}}$ is crystalline and $\rho_v|_{G_{F_{1,u}}} \sim \psi_1^{(u)} \oplus \cdots \oplus \psi_n^{(u)}$ with each $\psi_i^{(u)}$ a crystalline character.

If $u|\widetilde{v}|l$ with $v \in S$, then for $i = 1, \ldots, n$, we define $\psi_i^{(cu)} : G_{F_{1,cu}} \to \overline{\mathbb{Q}}_l^\times$ by $(\psi_i^{(cu)})^c \psi_i^{(u)} = \mu|_{G_{F_{1,u}}}$.

By Proposition 3.3.1 there is a finite, Galois, CM extension F_2/F_1 linearly disjoint from $F_1\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F_1 and a RAECSDC automorphic representation (π_2, χ_2) of $GL_n(\mathbb{A}_{F_2})$ such that

- $\overline{r}_{l,\imath}(\pi_2) \cong \breve{\overline{r}}|_{G_{F_2}}$;
- $r_{l,\imath}(\chi_2) = \widetilde{\mu}|_{G_{F_2^+}} \omega_l^{(n-1)m} \epsilon_l^{(1-n)(m-1)}$;
- π_2 is $\imath$-ordinary and unramified above l;
- if $\tau : F_2 \hookrightarrow \overline{\mathbb{Q}}_l$, then $\mathrm{HT}_\tau(r_{l,\imath}(\pi_2)) = \{0, m, 2m, \dots, (n-1)m\}$;
- π_2 is unramified outside S;
- and if $v \not| l$ is in S and if u is a prime of F_2 above $\widetilde{v}$ then $r_{l,\imath}(\pi_2)|_{G_{F_{2,u}}} \sim \rho_v|_{G_{F_{2,u}}}$.

In particular if $u|l$ is a place of F_2 then

$$r_{l,\imath}(\pi_2)|_{G_{F_{2,u}}} \sim 1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}.$$

Choose a CM extension M/F_2 such that

- M/F_2 is cyclic of degree n;
- M is linearly disjoint from $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over F;
- and all primes of F_2 above l split completely in M.

Choose a prime u_q of F_2 above a rational prime q such that

- $q \neq l$ and q splits completely in M;
- $\overline{r}$ is unramified above q.

If $v|ql$ is a prime of F_2 we label the primes of M above v as $v_{M,1}, \dots, v_{M,n}$ so that $(cv)_{M,i} = c(v_{M,i})$. Choose continuous characters

$$\theta, \theta' : G_M \longrightarrow \overline{\mathbb{Q}}_l^\times$$

such that

- the reductions $\overline{\theta}$ and $\overline{\theta}'$ are equal;
- $\theta\theta^c = r_{l,\imath}(\chi_2)\epsilon_l^{1-n}$ and $\theta'(\theta')^c = \mu$;
- θ and θ' are de Rham;
- if $\tau : M \hookrightarrow \overline{\mathbb{Q}}_l$ lies above a place $v_{M,i}|l$ of M then $\mathrm{HT}_\tau(\theta) = \{(i-1)m\}$ and $\mathrm{HT}_\tau(\theta') = \mathrm{HT}_{\tau|_{F_1}}(\psi_i^{(v_{M,i}|_{F_1})})$;
- θ and θ' are unramified at $u_{q,M,i}$ for $i > 1$, but q divides $\#\theta(I_{M_{u_{q,M,1}}})$ and $\#\theta'(I_{M_{u_{q,M,1}}})$.

(Use Lemma 4.1.6 of [CHT08] twice.)

Note the following:

- If $u|l$ is a place of F_2 and if $K/F_{2,u}$ is a finite extension over which θ and θ' become crystalline and $\overline{\theta} = \overline{\theta}'$ become trivial, then

$$(\mathrm{Ind}\,_{G_M}^{G_{F_2}} \theta)|_{G_K} \sim 1 \oplus \epsilon_l^{-m} \oplus \cdots \oplus \epsilon_l^{(1-n)m}$$

  and

$$(\mathrm{Ind}\,_{G_M}^{G_{F_2}} \theta')|_{G_K} \sim \psi_1^{(u|_{F_1})}|_{G_K} \oplus \cdots \oplus \psi_n^{(u|_{F_1})}|_{G_K}.$$

- $(\mathrm{Ind}\,_{G_M}^{G_{F_2}} \theta)^c \cong (\mathrm{Ind}\,_{G_M}^{G_{F_2}} \theta)^\vee \otimes r_{l,\imath}(\chi_2)\epsilon_l^{1-n}$ and $(\mathrm{Ind}\,_{G_M}^{G_{F_2}} \theta')^c \cong (\mathrm{Ind}\,_{G_M}^{G_{F_2}} \theta')^\vee \otimes \mu$.

- The representation
$$\breve{\overline{r}}|_{G_{F_2(\zeta_l)}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_2}} \overline{\theta})|_{G_{F_2(\zeta_l)}} \cong \overline{r}_{l,\imath}(\pi_2)|_{G_{F_2(\zeta_l)}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_2}} \overline{\theta}')|_{G_{F_2(\zeta_l)}}$$
is irreducible, and hence by Proposition 2.1.2
$$(\breve{\overline{r}}|_{G_{F_2}} \otimes (\mathrm{Ind}_{G_M}^{G_{F_2}} \overline{\theta}))(G_{F_2(\zeta_l)})$$
is adequate.
[As $\breve{\overline{r}}|_{G_{M(\zeta_l)}}$ is irreducible, we see that the restriction to $G_{M(\zeta_l)}$ of any constituent of $(\breve{\overline{r}}|_{G_{F_2}} \otimes \mathrm{Ind}_{G_M}^{G_{F_2}} \overline{\theta})|_{G_{F_2(\zeta_l)}}$ is a sum of $\breve{\overline{r}}|_{G_{M(\zeta_l)}} \overline{\theta}^{\tau}|_{G_{M(\zeta_l)}}$ as τ runs over some subset of $\mathrm{Gal}\,(M/F_2)$. Looking at ramification above u_q we see that the $\breve{\overline{r}}|_{G_{M(\zeta_l)}} \overline{\theta}^{\tau}|_{G_{M(\zeta_l)}}$ are permuted transitively by $\mathrm{Gal}\,(M/F_2)$ and hence $(\breve{\overline{r}}|_{G_{F_2}} \otimes \mathrm{Ind}_{G_M}^{G_{F_2}} \overline{\theta})|_{G_{F_2(\zeta_l)}}$ is irreducible.]

Let F_3/F_2 be a finite, soluble, Galois, CM extension linearly disjoint from $\overline{F}_2^{\ker \mathrm{Ind}_{G_M}^{G_{F_2}} \overline{\theta}} \overline{F}^{\ker \overline{r}}(\zeta_l)$ over F_2 such that

- $\theta|_{G_{F_3M}}$ and $\theta'|_{G_{F_3M}}$ are crystalline above l and unramified away from l;
- MF_3/F_3 is unramified everywhere.

Then there is a RAECSDC automorphic representation (π_3, χ_3) of $GL_{n^2}(\mathbb{A}_{F_3})$ such that

- $r_{l,\imath}(\pi_3) \cong (r_{l,\imath}(\pi_2) \otimes \mathrm{Ind}_{G_M}^{G_{F_2}} \theta')|_{G_{F_3}}$;
- $r_{l,\imath}(\chi_3) = \mu r_{l,\imath}(\chi_2)\epsilon_l^{n(n-1)}\delta_{F_3/F_3^+}$;
- π_3 is unramified above l and outside S.

[The representation π_3 is the automorphic induction of $(\pi_2)_{MF_3} \otimes (\phi' |\ |^{n(n-1)/2} \circ \det)$ to F_3, where $r_{l,\imath}(\phi') = \theta'|_{G_{F_3M}}$. The first two properties are clear. The third property follows by the choice of F_3 and the fact that π_2 is unramified above l and outside S.]

Let $\widetilde{S}$ denote the set of $\widetilde{v}$ as v runs over S, let S_3 denote the primes of F_3^+ above S and $\widetilde{S}_3$ the primes of F_3 above $\widetilde{S}$. If $v \in S_3$, let $\widetilde{v}$ denote the element of $\widetilde{S}_3$ lying above it. For $v \in S_3$ with $v \not| l$ (resp. $v|l$) let $\mathcal{C}_{3,v}$ denote the unique component of $R^{\square}_{\overline{r}_{l,\imath}(\pi_3)|_{G_{F_{3,\widetilde{v}}}}} \otimes \overline{\mathbb{Q}}_l$ (resp. $R^{\square}_{\overline{r}_{l,\imath}(\pi_3)|_{G_{F_{3,\widetilde{v}}}}, \{\mathrm{HT}_\tau(r_{l,\imath}(\pi_3)|_{G_{F_{3,\widetilde{v}}}})\}, \mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$) containing $r_{l,\imath}(\pi_3)|_{G_{F_{3,\widetilde{v}}}}$. Choose a finite extension $L/\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ such that

- L contains the image of each embedding $F_3 \hookrightarrow \overline{\mathbb{Q}}_l$;
- L contains the image of μ and of θ;
- $r_{l,\imath}(\pi_3)$ is defined over L;
- each of the components $\mathcal{C}_v$ for $v \in S$ and $\mathcal{C}_{3,v}$ for $v \in S_3$ is defined over L.

Set
$$s = \mathrm{Ind}_{G_{M^+}, G_M}^{G_{F_2^+}, G_{F_2}, r_{l,\imath}(\chi_2)\epsilon_l^{1-n}} (\theta, r_{l,\imath}(\chi_2)\epsilon_l^{1-n}) : G_{F_2^+} \longrightarrow \mathcal{G}_n(\mathcal{O}_L)$$
in the notation of section 1.1 of this paper and section 2.1 of [CHT08]. Thus $\nu \circ s = r_{l,\imath}(\chi_2)\epsilon_l^{1-n}$. For $v \in S$ (resp. $v \in S_3$) let $\mathcal{D}_v$ (resp. $\mathcal{D}_{3,v}$) denote the deformation problem for $\overline{r}|_{G_{F_{\widetilde{v}}}}$ (resp. $\overline{r}_{l,\imath}(\pi_3)|_{G_{F_{3,\widetilde{v}}}}$) over $\mathcal{O}_L$ corresponding to $\mathcal{C}_v$ (resp. $\mathcal{C}_{3,v}$). Also let
$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}_L, \overline{r}, \mu, \{\mathcal{D}_v\})$$

and

$$\mathcal{S}_3 = (F_3/F_3^+, S_3, \widetilde{S}_3, \mathcal{O}_L, \overline{r}_{l,\imath}(\pi_3), \mu r_{l,\imath}(\chi_2)\epsilon_l^{1-n}\delta_{F_3^+/F_3}, \{\mathcal{D}_{3,v}\}).$$

There is a natural map

$$R^{\mathrm{univ}}_{\mathcal{S}_3} \longrightarrow R^{\mathrm{univ}}_{\mathcal{S}}$$

induced by $r^{\mathrm{univ}}_{\mathcal{S}}|_{G_{F_3^+}} \otimes s|_{G_{F_3^+}}$. [We must check that if $u \in S_3$ then $\breve{r}^{\mathrm{univ}}_{\mathcal{S}}|_{G_{F_{3,\widetilde{u}}}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_{F_{3,\widetilde{u}}}} \in \mathcal{D}_{3,u}$. Let $v = u|_{F^+}$ and let $\rho^{\square}_{v,\mathcal{C}_v}$ denote the universal lift of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ to $R^{\square}_{\mathcal{O}_L,\overline{r}|_{G_{F_{\widetilde{v}}}},\mathcal{C}_v}$. It suffices to show that $\rho^{\square}_{v,\mathcal{C}_v}|_{G_{F_{3,\widetilde{u}}}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_{F_{3,\widetilde{u}}}} \in \mathcal{D}_{3,u}$. For this, it suffices to show if $\rho : G_{F_{\widetilde{v}}} \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ is a lift of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ lying on $\mathcal{C}_v$, then $\rho|_{G_{F_{3,\widetilde{u}}}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_{F_{3,\widetilde{u}}}}$ lies on $\mathcal{C}_{3,u}$. If $u|l$, then $\rho|_{G_{F_{3,\widetilde{u}}}} \sim (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')|_{G_{F_{3,\widetilde{u}}}}$ and $(\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_{F_{3,\widetilde{u}}}} \sim r_{l,\imath}(\pi_2)|_{G_{F_{3,\widetilde{u}}}}$ and hence

$$\rho|_{G_{F_{3,\widetilde{u}}}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_{F_{3,\widetilde{u}}}} \sim (r_{l,\imath}(\pi_2) \otimes \mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')|_{G_{F_{3,\widetilde{u}}}} \cong r_{l,\imath}(\pi_3)|_{G_{F_{3,\widetilde{u}}}}.$$

If $u \not| l$, note that $\rho|_{G_{F_{3,\widetilde{u}}}} \sim r_{l,\imath}(\pi_2)|_{G_{F_{3,\widetilde{u}}}}$ (since ρ_v is robustly smooth, we have $\rho_v|_{G_{F_{3,\widetilde{u}}}} \rightsquigarrow \rho|_{G_{F_{3,\widetilde{u}}}}$ and $\rho_v|_{G_{F_{3,\widetilde{u}}}} \rightsquigarrow r_{l,\imath}(\pi_2)|_{G_{F_{3,\widetilde{u}}}}$). By the choice of F_3 we have $(\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_{F_{3,\widetilde{u}}}} \sim (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')|_{G_{F_{3,\widetilde{u}}}}$. Hence

$$\rho|_{G_{F_{3,\widetilde{u}}}} \otimes (\mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta)|_{G_{F_{3,\widetilde{u}}}} \sim (r_{l,\imath}(\pi_2) \otimes \mathrm{Ind}\,_{G_M}^{G_{F_2}}\theta')|_{G_{F_{3,\widetilde{u}}}} \cong r_{l,\imath}(\pi_3)|_{G_{F_{3,\widetilde{u}}}}$$

and we are done.] It follows from Lemma 1.2.2 that this map makes $R^{\mathrm{univ}}_{\mathcal{S}}$ a finite $R^{\mathrm{univ}}_{\mathcal{S}_3}$-module. By Theorem 2.2.2 $R^{\mathrm{univ}}_{\mathcal{S}_3}$ is a finite $\mathcal{O}_L$-module, and hence $R^{\mathrm{univ}}_{\mathcal{S}}$ is a finite $\mathcal{O}_L$-module. On the other hand by Proposition 1.5.1 $R^{\mathrm{univ}}_{\mathcal{S}}$ has Krull dimension at least 1. Hence $\mathrm{Spec}\, R^{\mathrm{univ}}_{\mathcal{S}}$ has a $\overline{\mathbb{Q}}_l$-point. This point gives rise to the desired lifting of $\overline{r}$. □

4.4. **Change of weight and level.**

In this section we combine Theorems 4.2.1 and 4.3.1 to obtain a general "change of weight and level" result in the potentially diagonalizable case.

Theorem 4.4.1. *Let F be an imaginary CM field with maximal totally real subfield F^+. Let $n \in \mathbb{Z}_{\geq 1}$ be an integer, and let $l > 2(n+1)$ be an odd prime, such that $\zeta_l \notin F$ and all primes of F^+ above l split in F. Let S be a finite set of finite places of F^+, including all places above l, such that each place in S splits in F. For each place $v \in S$ choose a place $\widetilde{v}$ of F lying over v.*

Let (π, χ) be a RAECSDC automorphic representation of $GL_n(\mathbb{A}_F)$ unramified outside S and such that $\overline{r}_{l,\imath}(\pi)|_{G_{F(\zeta_l)}}$ is irreducible. Suppose further that

- *either π is $\imath$-ordinary,*
- *or π has level potentially prime to l and $r_{l,\imath}(\pi)|_{G_{F_v}}$ is potentially diagonalizable for all $v|l$.*

Let $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^{\times}$ be a continuous, de Rham character unramified outside S such that $\overline{\mu} = \overline{r}_{l,\imath}(\chi)\overline{\epsilon}_l^{1-n}$. For $v \in S$ let $\rho_v : G_{F_{\widetilde{v}}} \to GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ be a lift of $\overline{r}|_{G_{F_{\widetilde{v}}}}$. If $v|l$, assume further that ρ_v is potentially diagonalizable, and that for all $\tau : F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$, $\mathrm{HT}_\tau(\rho_v)$ consists of n distinct integers.

Then there is a RAECSDC automorphic representation (π', χ') of $GL_n(\mathbb{A}_F)$ such that

(1) $\bar{r}_{l,\imath}(\pi') \cong \bar{r}_{l,\imath}(\pi)$;
(2) $r_{l,\imath}(\chi')\epsilon_l^{1-n} = \mu$;
(3) π' *has level potentially prime to* l;
(4) π' *is unramified outside* S;
(5) *for* $v \in S$ *we have* $\rho_v \sim r_{l,\imath}(\pi')|_{G_{F_{\widetilde{v}}}}$.

Proof. By Theorem 4.3.1 there is a continuous homomorphism

$$r : G_{F^+} \to \mathcal{G}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

such that

- $\bar{\breve{r}} \cong \bar{r}_{l,\imath}(\pi)$;
- $\breve{r}$ is unramified outside S;
- $\nu \circ r = \mu$;
- if $v|l$ then $\breve{r}|_{G_{F_{\widetilde{v}}}}$ is potentially crystalline;
- if $v \in S$ then $\breve{r}|_{G_{F_{\widetilde{v}}}} \sim \rho_v$.

By Theorem 4.2.1 r is automorphic of level potentially prime to l and our present theorem follows (using local-global compatibility). $\square$

4.5. **Potential automorphy II.**

We can now turn to our main potential automorphy theorem for single l-adic representations.

Theorem 4.5.1. *Suppose that we are in the following situation.*

(a) *Let* F/F_0 *be a finite, Galois extension of CM fields; and let* F^+ *and* F_0^+ *denote their maximal totally real subfields.*
(b) *Let* $\mathcal{I}$ *be a finite set.*
(c) *For each* $i \in \mathcal{I}$ *let* n_i *and* d_i *be positive integers and* l_i *be an odd rational prime such that* $l_i \geq 2(d_i+1)$ *and* $\zeta_{l_i} \notin F$. *Also choose* $\imath_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$ *for each* $i \in \mathcal{I}$.
(d) *For each* $i \in \mathcal{I}$ *let* $\mu_i : G_{F^+} \to \overline{\mathbb{Q}}_{l_i}^\times$ *be a continuous character.*
(e) *For each* $i \in \mathcal{I}$ *let* $r_i : G_F \to GL_{n_i}(\overline{\mathbb{Q}}_{l_i})$ *be a continuous representation, such that* d_i *is the maximum dimension of an irreducible constituent of the restriction of* $\bar{r}_i$ *to the closed subgroup of* G_F *generated by all Sylow pro-*l_i*-subgroups.*
(f) *Let* $F^{(\mathrm{avoid})}/F$ *be a finite Galois extension.*

Suppose moreover that the following conditions are satisfied for every $i \in \mathcal{I}$.

(1) **(Being unramified almost everywhere)** r_i *is unramified at all but finitely many primes.*
(2) **(Odd essentially self-duality)** μ_i *is totally odd and* (r_i, μ_i) *is essentially conjugate self-dual.*
(3) **(Potential diagonalizability and regularity at primes above** l_i**)** r_i *is potentially diagonalizable (and hence potentially crystalline) at each prime* v *of* F^+ *above* l_i *and for each* $\tau : F \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ *the multiset* $\mathrm{HT}_\tau(r_i)$ *contains* n_i *distinct elements.*
(4) **(Irreducibility)** $\bar{r}_i|_{G_{F(\zeta_{l_i})}}$ *is irreducible.*

Then we can find a finite CM extension F'/F *and for each* $i \in \mathcal{I}$ *a RAECSDC automorphic representation* (π_i, χ_i) *of* $GL_{n_i}(\mathbb{A}_{F'})$ *such that*

(i) *F'/F_0 is Galois,*
(ii) *F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F,*
(iii) *π_i is unramified above l_i, and*
(iv) $(r_{l_i,\imath_i}(\pi_i), r_{l_i,\imath_i}(\chi_i)\epsilon_{l_i}^{1-n_i}) \cong (r_i|_{G_{F'}}, \mu_i|_{G_{(F')^+}})$.

We remark that by Lemma 1.4.2 the hypothesis of potential diagonalizability will hold if l_i is unramified in F^+, and r_i is crystalline at all primes $v|l_i$, and $\mathrm{HT}_\tau(r_i)$ is contained in an interval of the form $[a_\tau, a_\tau + l - 2]$ for all τ (the "Fontaine-Laffaille" case).

Proof. By Proposition 3.3.1 there is a finite CM extension F'/F and RAECSDC automorphic representation (π_i', χ_i') of $GL_{n_i}(\mathbb{A}_{F'})$ such that

- F'/F_0 is Galois;
- F' is linearly disjoint from $\overline{F}^{\bigcap_i \ker \overline{r}_i} F^{(\mathrm{avoid})}(\zeta_{\prod_i l_i})$ over F;
- $\overline{r}_{l_i,\imath_i}(\pi_i') \cong \overline{r}_i|_{G_{F'}}$;
- $r_{l_i,\imath_i}(\chi_i')\epsilon_{l_i}^{1-n_i} = \mu_i|_{G_{(F')^+}}$;
- π_i' is unramified above l;
- π_i is $\imath_i$-ordinary.

Then the current theorem follows from Theorem 4.2.1. □

We can immediately deduce a version over totally real fields. For instance we have the following.

Corollary 4.5.2. *Suppose F^+ is a totally real field and $n \in \mathbb{Z}_{\geq 1}$. Suppose that $l \geq 2(n+1)$ is a rational prime.*

Suppose also that
$$r : G_{F^+} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$
is an continuous representation and that
$$\mu : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_l^\times$$
is a continuous character. Let $\overline{r}$ denote the semi-simplification of the reduction of r. Suppose that the following conditions hold.

(1) **(Being unramified almost everywhere)** *r is unramified at all but finitely many primes.*
(2) **(Odd essential self-duality)** *(r, μ) is totally odd, essentially conjugate self-dual.*
(3) **(Potential diagonalizability and regularity at primes above l)** *r is potentially diagonalizable (and hence potentially crystalline) at each prime v of F^+ above l and for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ the multiset $\mathrm{HT}_\tau(r)$ contains n distinct elements.*
(4) **(Irreducibility)** *$\overline{r}|_{G_{F^+(\zeta_l)}}$ is irreducible.*

Then there is a Galois totally real extension $F^{+,\prime}/F^+$ such that $(r|_{G_{F^{+,\prime}}}, \mu|_{G_{F^{+,\prime}}})$ is automorphic of level prime to l.

Proof. Choose F/F^+ a totally imaginary quadratic extension in which all the places lying over l split completely, and which is linearly disjoint from $(\overline{F}^+)^{\ker \mathrm{ad}\, \overline{r}}(\zeta_l)$ over F^+. The representation $r|_{G_F}$ satisfies the hypotheses of Theorem 4.5.1, so that there is a finite Galois CM extension F'/F such that $(r|_{G_{F'}}, \mu|_{G_{F',+}})$ is automorphic of level prime to l. By Lemma 1.5 of [BLGHT09], $(r|_{G_{F',+}}, \mu|_{G_{F',+}})$ is also automorphic of level prime to l, as required. □

Combining Theorem 4.5.1 with Theorem 4.3.1 we get a potential modularity theorem for mod l Galois representations which strengthens Proposition 3.3.1.

Corollary 4.5.3. *Suppose that we are in the following situation.*

(a) *Let F/F_0 be a finite, Galois extension of CM fields, and let F^+ and F_0^+ denote their maximal totally real subfields. Choose a complex conjugation $c \in G_{F^+}$.*
(b) *Let $\mathcal{I}$ be a finite set.*
(c) *For each $i \in \mathcal{I}$ let n_i and d_i be positive integers and $l_i \geq 2(d_i+1)$ be an odd rational prime such that $\zeta_{l_i} \notin F$. Also choose $\imath_i : \overline{\mathbb{Q}}_{l_i} \xrightarrow{\sim} \mathbb{C}$ for each $i \in \mathcal{I}$.*
(d) *For each $i \in \mathcal{I}$ let $\mu_i : G_{F^+} \to \overline{\mathbb{Q}}_{l_i}^\times$ be a continuous, totally odd, de Rham character.*
(e) *For each $i \in \mathcal{I}$ let $\overline{r}_i : G_F \to GL_{n_i}(\overline{\mathbb{F}}_{l_i})$ be an irreducible continuous representation so that d_i is the maximal dimension of an irreducible constituent of the restriction of $\overline{r}_i$ to the closed subgroup of G_F generated by all Sylow pro-l_i-subgroups. Suppose also that $(\overline{r}_i, \overline{\mu}_i)$ is essentially conjugate self dual and that $\overline{r}_i|_{G_{F(\zeta_{l_i})}}$ is irreducible.*
(f) *Let S denote a finite* Gal (F/F^+)*-invariant set of primes of F including all those dividing $\prod_i l_i$ and all those at which some $\overline{r}_i$ ramifies.*
(g) *for each $i \in \mathcal{I}$ and $v \in S$ let $\rho_{i,v} : G_{F_v} \to GL_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_{l_i}})$ denote a lift of $\overline{r}_i|_{G_{F_v}}$ such that $\rho_{i,cv}^c \cong \mu_i|_{G_{F_v}} \rho_{i,v}^\vee$. If $v|l_i$ further assume that $\rho_{i,v}$ is potentially diagonalizable and that for each $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_{l_i}$ the set $\mathrm{HT}_\tau(\rho_{v,i})$ has n distinct elements.*
(h) *Let $F^{(\mathrm{avoid})}/F$ be a finite Galois extension.*

Then we can find a finite CM extension F'/F and for each $i \in \mathcal{I}$ a RAECSDC automorphic representation (π_i, χ_i) of $GL_{n_i}(\mathbb{A}_{F'})$ such that

(1) *F'/F_0 is Galois,*
(2) *F' is linearly disjoint from $F^{(\mathrm{avoid})}$ over F,*
(3) *$\overline{r}_{l_i,\imath_i}(\pi_i) \cong \overline{r}_i|_{G_{F'}}$;*
(4) *$r_{l_i,\imath_i}(\chi_i)\epsilon_{l_i}^{1-n_i} = \mu_i|_{G_{(F')^+}}$;*
(5) *π_i has level potentially prime to l_i;*
(6) *if u is a prime of F' not lying above a prime in S then $r_{l_i,\imath_i}(\pi_i)$ is unramified at u;*
(7) *if u is a prime of F' lying above an element $v \in S$ then $r_{l_i,\imath_i}(\pi_i)|_{G_{F'_u}} \sim \rho_v|_{G_{F'_u}}$.*

Proof. Note that $(\overline{r}_i, \overline{\mu}_i|_{G_F})$ extends to a continuous homomorphism $\tilde{\overline{r}}_i : G_{F^+} \to \mathcal{G}_{n_i}(\overline{\mathbb{F}}_{l_i})$ with $\nu \circ \tilde{\overline{r}}_i = \overline{\mu}_i$ (see section 1.1). As in the proof of Proposition 3.3.1, we may reduce to the case where all elements of S are split over F^+. Then by Theorem 4.3.1, we see that for each $i \in \mathcal{I}$ there exists a lift

$$r_i : G_F \to GL_{n_i}(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

of $\overline{r}_i$ such that

- $r_i^c \cong r_i^\vee \mu_i|_{G_F}$;

- if $v \in S$, then $r_i|_{G_{F_v}} \sim \rho_{i,v}$;
- r_i is unramified outside S.

The result now follows from Theorem 4.5.1. $\square$

5. Compatible systems.

5.1. **Compatible systems: definitions.**

Let F denote a number field. By a *rank n weakly compatible system of l-adic representations* $\mathcal{R}$ of G_F defined over M we shall mean a 5-tuple

$$(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

where

(1) M is a number field;
(2) S is a finite set of primes of F;
(3) for each prime $v \notin S$ of F, $Q_v(X)$ is a monic degree n polynomial in $M[X]$;
(4) for each prime λ of M (with residue characteristic l say)

$$r_\lambda : G_F \longrightarrow GL_n(\overline{M}_\lambda)$$

is a continuous, semi-simple, representation such that
- if $v \notin S$ and $v \not| l$ is a prime of F then r_λ is unramified at v and $r_\lambda(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$,
- while if $v|l$ then $r_\lambda|_{G_{F_v}}$ is de Rham and in the case $v \notin S$ crystalline;

(5) for $\tau : F \hookrightarrow \overline{M}$, H_τ is a multiset of n integers such that for any $\overline{M} \hookrightarrow \overline{M}_\lambda$ over M we have $\mathrm{HT}_\tau(r_\lambda) = H_\tau$.

We will refer to a rank 1 weakly compatible system of representations as a weakly compatible system of characters.

We make the following subsidiary definitions.

- We define the usual linear algebra and group theoretic operations on weakly compatible systems by applying the corresponding operation to each r_λ. For instance
$$\mathcal{R}^\vee = (M, S, \{X^n Q_v(0)^{-1} Q_v(X^{-1})\}, \{r_\lambda^\vee\}, \{-H_\tau\}),$$
where $-H_\tau = \{-h : h \in H_\tau\}$.
- We will call $\mathcal{R}$ *regular* if for each $\tau : F \hookrightarrow \overline{M}$ every element of H_τ has multiplicity 1.
- We will call $\mathcal{R}$ *extremely regular* if it is regular and for some $\tau : F \hookrightarrow \overline{M}$ the multiset H_τ has the following property: if H and H' are subsets of H_τ of the same cardinality and if $\sum_{h \in H} h = \sum_{h \in H'} h$ then $H = H'$.
- If F is totally real and if $n = 1$ then we will call $\mathcal{R}$ totally odd if for some place λ of M we have $r_\lambda(c_v) = -1$ for all infinite places v of F. In this case this will also be true for all places λ of M.
- If F is CM or totally real and if $\mathcal{M} = (M, S, \{X - \alpha_v\}, \{\mu_\lambda\}, \{w\})$ is a weakly compatible system of characters of G_{F^+} then we will call $(\mathcal{R}, \mathcal{M})$ *essentially conjugate self-dual* if for all primes λ of M the pair (r_λ, μ_λ) is essentially conjugate self-dual.
- We will call $\mathcal{R}$ *totally odd, essentially conjugate self-dual* if F is CM (or totally real) and if (perhaps after extending M) there is a weakly compatible system of characters $\mathcal{M} = (M, S, \{X - \alpha_v\}, \{\mu_\lambda\}, \{w\})$ of G_{F^+} such that for all primes λ of M the pair (r_λ, μ_λ) is totally odd, essentially self-dual.
- We will call $\mathcal{R}$ *irreducible* if there is a set $\mathcal{L}$ of rational primes of Dirichlet density 1 such that for $\lambda | l \in \mathcal{L}$ the representation r_λ is irreducible.

- We will call $\mathcal{R}$ *strictly compatible* if for each finite place v of F there is a Weil-Deligne representation $\mathrm{WD}_v(\mathcal{R})$ of W_{F_v} over $\overline{M}$ such that for each place λ of M not dividing the residue characteristic of v and every M-linear embedding $\varsigma : \overline{M} \hookrightarrow \overline{M}_\lambda$ the push forward $\varsigma \mathrm{WD}_v(\mathcal{R}) \cong \mathrm{WD}(r_\lambda|_{G_{F_v}})^{\text{F-ss}}$.
- We will call $\mathcal{R}$ *pure* of weight w if
  - for each $v \notin S$, each root α of $Q_v(X)$ in $\overline{M}$ and each $\imath : \overline{M} \hookrightarrow \mathbb{C}$ we have
  $$|\imath\alpha|^2 = (\#k(v))^w;$$
  - and for each $\tau : F \hookrightarrow \overline{M}$ and each complex conjugation c in $\mathrm{Gal}\,(\overline{M}/\mathbb{Q})$ we have
  $$H_{c\tau} = \{w - h : \ \ h \in H_\tau\}.$$
- We will call a $\mathcal{R}$ *strictly pure* of weight w if
  - $\mathcal{R}$ is strictly compatible and for each prime v of F the Weil-Deligne representation $\mathrm{WD}_v(\mathcal{R})$ is pure of weight w in the sense of [TY07] (see the paragraph before Lemma 1.4 of that paper);
  - and for each $\tau : F \hookrightarrow \overline{M}$ and each complex conjugation c in $\mathrm{Gal}\,(\overline{M}/\mathbb{Q})$ we have
  $$H_{c\tau} = \{w - h : \ \ h \in H_\tau\}.$$
- If $\imath : M \hookrightarrow \mathbb{C}$ we define the partial L-function
  $$L^S(\imath\mathcal{R}, s) = \prod_{v \notin S} (\#k(v))^{ns} / \imath Q_v((\#k(v))^s).$$

  This may or may not converge. If $\mathcal{R}$ is pure of weight w then it will converge to an analytic function in $\mathrm{Re}\, s > 1 + w/2$. If $\lambda | l$ and every place of F above l lies in S, then $L^S(\imath\mathcal{R}, s)$ depends only on r_λ so, if $\widetilde{\imath} : \overline{M}_\lambda \xrightarrow{\sim} \mathbb{C}$ extends $\imath$, we will sometimes write $L^S(\widetilde{\imath} r_\lambda, s)$ instead of $L^S(\imath\mathcal{R}, s)$. This makes sense even for r_λ not part of a weakly compatible system, provided that S contains all primes above l and all primes at which r_λ ramifies.

  If $\mathcal{R}$ is strictly compatible then we can define the L-function
  $$L(\imath\mathcal{R}, s) = \prod_{v \not| \infty} L(\imath \mathrm{WD}_v(\mathcal{R}), s)$$
  which differs from $L^S(\imath\mathcal{R}, s)$ only by the addition of finitely many Euler factors.
- Suppose that $\mathcal{R}$ is strictly compatible, pure of weight w and regular. Also let $\psi = \prod_v \psi_v : \mathbb{A}_F/F \to \mathbb{C}^\times$ be the non-trivial additive character such that if v is real then $\psi_v(x) = e^{2\pi i x}$; if v is complex then $\psi_v(x) = e^{2\pi i (x + c_v x)}$; while if v is p-adic then $\psi_v(x) = \psi_p(\mathrm{tr}\,_{F_v/\mathbb{Q}_p}(x))$ where $\psi_p|_{\mathbb{Z}_p} = 1$ and $\psi_p(1/p) = e^{-2\pi i/p}$. Write $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(s+1)$.

  If v is an infinite place of F define $L(\mathcal{R}|_{G_{F_v}}, w, s)$ to be

$$\begin{array}{ll} \Gamma_\mathbb{C}(s - w/2)^n & \text{if } v \text{ is complex} \\ \Gamma_\mathbb{C}(s - w/2)^{n/2} & \text{if } v \text{ is real and } n \text{ even} \\ \Gamma_\mathbb{C}(s - w/2)^{(n-1)/2}\Gamma_\mathbb{R}(s - (w - 1 + (-1)^{w/2}(\det \mathcal{R}(c_v)))/2) & \text{if } v \text{ is real and } n \text{ odd,} \end{array}$$

and define $\epsilon(\mathcal{R}|_{G_{F_v}}, w, \psi_v, s)$ to be

$$\begin{array}{ll} 1 & \text{if } v \text{ is complex} \\ i^{n/2} & \text{if } v \text{ is real and } n \text{ even} \\ i^{(n-(-1)^{w/2}(\det \mathcal{R}(c_v)))/2} & \text{if } v \text{ is real and } n \text{ odd.} \end{array}$$

(As $\{\det r_\lambda\}$ is a weakly compatible system of characters, $\det r_\lambda(c_v) = \pm 1$ is independent of λ. We set $\det \mathcal{R}(c_v) = \det r_\lambda(c_v) \in \{\pm 1\}$. Also note that by purity if n is odd and v real then w is even.)

Define

$$L(\{H_\tau\}, s) = \sqrt{2\pi}^{\sum_{\tau:F\hookrightarrow \overline{M}} \sum_{h\in H_\tau} |h-w/2|}$$
$$\left(\prod_{\tau:F\hookrightarrow \overline{M}} \prod_{h\in H_\tau} (s-w/2)(s+1-w/2)\ldots(s+|h-w/2|-1-w/2)\right)^{1/2}$$

and

$$\epsilon(\{H_\tau\}, \psi, s) = i^{\sum_{\tau:F\hookrightarrow \overline{M}} \sum_{h\in H_\tau} |h-w/2|}.$$

(Note that by purity each factor $(s+i-w/2)$ occurs twice. Thus we define the square root to be the corresponding product of linear factors where each factor occurs half as often. Also note that by purity $\sum_{\tau:F\hookrightarrow \overline{M}} \sum_{m\in H_\tau} |m-w/2| \in \mathbb{Z}$.)

Finally we define the completed L-function

$$\Lambda(\imath\mathcal{R}, s) = L(\imath\mathcal{R}, s)\left(\prod_{v|\infty} L(\mathcal{R}|_{G_{F_v}}, w, s)\right) L(\{H_\tau\}, s)$$

and the epsilon factor

$$\epsilon(\imath\mathcal{R}, s) = \left(\prod_{v\not|\infty} \epsilon(\imath \mathrm{WD}_v(\mathcal{R}), \psi_v, s)\right)\left(\prod_{v|\infty} \epsilon(\mathcal{R}|_{G_{F_v}}, w, \psi_v, s)\right)\epsilon(\{H_\tau\}, \psi, s).$$

We remark that while the way we have chosen to phrase these definitions may seem slightly non-standard, the final definitions of $\Lambda(\imath\mathcal{R}, s)$ and $\epsilon(\imath\mathcal{R}, s)$ are the usual ones.

- We will call $\mathcal{R}$ *automorphic* if there is a regular, algebraic, cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$ and an embedding $\imath : M \hookrightarrow \mathbb{C}$, such that if $v \notin S$ then π_v is unramified and $\mathrm{rec}(\pi_v|\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\imath(Q_v(X))$. Note that in this case, for any embedding $\imath' : M \hookrightarrow \mathbb{C}$ there is a regular, algebraic, cuspidal automorphic representation π' of $GL_n(\mathbb{A}_F)$, such that if $v \notin S$ then π'_v is unramified and $\mathrm{rec}(\pi'_v|\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\imath'(Q_v(X))$. Also note that if $\mathcal{R}$ is totally odd, essentially conjugate self-dual then π (and π') is RAECSDC or RAESDC.
- If F'/F is a finite extension we define $\mathcal{R}|_{G_{F'}}$ to be the weakly compatible system of representations of $G_{F'}$:

$$(M, S^{(F')}, \{Q_v^{(F')}(X)\}, \{r_\lambda|_{G_{F'}}\}, \{H_\tau^{(F')}\}),$$

where
  - $S^{(F')}$ is the set of primes of F' above S;
  - $H_\tau^{(F')} = H_{\tau|_F}$;

– $Q_v^{(F')}(X)$ is the monic polynomial in $M[X]$ of degree n with roots $\alpha^{[k(v):k(v|_F)]}$ as α runs over roots of $Q_{v|_F}(X)$.

We remark that if (π, χ) is a RAECSDC or RAESDC automorphic representation of $GL_n(\mathbb{A}_F)$ then $\{r_{l,\imath}(\chi)\}$ is a strictly pure compatible system of some necessarily even weight (because F^+ is totally real), which we will write $2(w+1-n)$. Then Caraiani has proved that $\{r_{l,\imath}(\pi)\}$ is a strictly pure compatible system of weight w. (See [Car10].)

5.2. **Compatible systems: lemmas.**

Lemma 5.2.1. *Suppose that $\mathcal{R}$ is a weakly compatible system of l-adic representations of G_F. Let G_λ denote the Zariski closure of $r_\lambda(G_F)$ in $GL_n/\overline{M}_\lambda$ and let G_λ^0 denote its connected component. Then there is a finite Galois extension F^1/F such that for all λ the map $G_F \to G_\lambda(\overline{M}_\lambda)$ induces an isomorphism* $\mathrm{Gal}\,(F^1/F) \stackrel{\sim}{\to} G_\lambda/G_\lambda^0$.

Proof. The proof is the same as the proof of Proposition 6.14 of [LP92]. (One must replace $\mathcal{G}$ by G_F, and ρ_l by r_λ, and G_l^0 by G_λ^0; and $\mathbb{Q}_l$ by $\overline{M}_\lambda$, and $\mathbb{Q}$ by M in section 6 of [LP92]. These arguments are due to Serre.) $\square$

Proposition 5.2.2. *Suppose that $\mathcal{R}$ is a regular, weakly compatible system of l-adic representations of G_F defined over M. If s is a sub-representation of r_λ then we will write $\overline{s}$ for the semi-simplification of the reduction of s. Also write l for the rational prime below λ. Then there is a set of rational primes $\mathcal{L}$ of Dirichlet density 1 such that if s is an irreducible sub-representation of r_λ with λ dividing an element of $\mathcal{L}$ then $\overline{s}|_{G_{F(\zeta_l)}}$ is irreducible.*

Proof. Let G_λ denote the Zariski closure of $r_\lambda(G_F)$ in $GL_n/\overline{M}_\lambda$, and let G_λ^0 denote its connected component. Let F^1 be the field defined in Lemma 5.2.1. By the argument of Proposition VII.1.8 and Lemma I.2.2 of [HT01] the regularity assumption tells us that G_λ^0 contains an element g whose characteristic polynomial has n distinct roots. As the images of Frobenius elements at primes which split completely in F^1/F are Zariski dense in G_λ^0 we conclude that infinitely many $Q_v(X)$, for v splitting completely in F^1/F, have n distinct roots. Replacing M by the splitting field of the product $Q_v(X)Q_{v'}(X)$ for two such v, v' with distinct residue characteristic, we may suppose that for all λ the image $r_\lambda(G_{F^1})$ contains an element with n distinct M_λ-rational eigenvalues. This implies that, perhaps after replacing each r_λ by a conjugate, we may suppose that $r_\lambda : G_F \to GL_n(\mathcal{O}_{M,\lambda})$ for all λ. If H is an open subgroup of G_F, then the Zariski closure of $r_\lambda(H)$ contains G_λ^0 and hence equals the Zariski closure of $r_\lambda(HG_{F^1})$. Thus the irreducible sub-representations of $r_\lambda|_H$ equal the irreducible sub-representations of $r_\lambda|_{HG_{F^1}}$. Further we see that these irreducible sub-representations are all defined over M_λ and have multiplicity 1 in r_λ. Again extending M if need be, we may also suppose that $M/\mathbb{Q}$ is Galois.

For a rational prime l define

$$r_l = \epsilon_l \oplus \bigoplus_{\lambda|l} r_\lambda : G_F \longrightarrow GL_{1+n[M:\mathbb{Q}]}(\mathbb{Q}_l).$$

We will introduce the following notations and observations.

- Let G_l denote the Zariski closure of the image of r_l. Write V_l for the G_l-module $\mathbb{Q}_l^{1+n[M:\mathbb{Q}]}$. The image Γ_l of G_F in $G_l(\mathbb{Q}_l)$ is open and compact. (See [Bog80].) The representations ϵ_l and r_λ for $\lambda|l$ can also be thought of as representations of G_l.
- Let G_l^0 denote the connected component of G_l (a reductive group over $\mathbb{Q}_l$) and write Γ_l^0 for $\Gamma_l \cap G_l^0(\mathbb{Q}_l)$. Then $F^0 = \overline{F}^{\ker(G_F \to \Gamma_l/\Gamma_l^0)}$ is independent of l. (See Proposition 6.14 of [LP92].)
- Let G_l^{ad} (resp. G_l^{der}) denote the adjoint (derived) group of G_l^0.
- Let Z_l denote the connected centre of G_l^0 and let C_l denote G_l^0/G_l^{der}. Note that ϵ_l factors through C_l. Write C_l^1 and Z_l^1 for the kernel of ϵ_l on C_l and Z_l . We have exact sequences
$$(0) \to C_l^1 \to C_l \stackrel{\epsilon_l}{\to} \mathbb{G}_m \to (0)$$
and
$$(0) \to Z_l^1 \to Z_l \stackrel{\epsilon_l}{\to} \mathbb{G}_m \to (0).$$
Again by [Bog80] we know that the image of G_{F^0} in $C_l(\mathbb{Q}_l)$ is open, and as C_l is abelian we conclude that the image of the product of the inertia groups at the primes of F^0 above l is open in $C_l(\mathbb{Q}_l)$.
- Let G_l^{SC} denote the simply connected cover of G_l^{ad}, so that we have surjective maps with finite central kernels $G_l^{\mathrm{SC}} \to G_l^{\mathrm{der}} \to G_l^{\mathrm{ad}}$. There is an integer $A \in \mathbb{Z}_{>0}$ such that $\#\ker(G_l^{\mathrm{SC}} \to G_l^{\mathrm{ad}})|A$ for all l. (Because $\dim G_l^{\mathrm{SC}}$ is bounded independently of l, e.g. by $(1+n[M:\mathbb{Q}])^2$.)
- Set $H_l = G_l^{\mathrm{SC}} \times Z_l$ and $H_l^1 = G_l^{\mathrm{SC}} \times Z_l^1$. Then there is a surjective map $H_l \to G_l^0$ with central kernel of order dividing A. (The kernel equals $\ker(G_l^{\mathrm{SC}} \to G_l^{\mathrm{ad}})$.) Then the cokernel of the map $H_l(\mathbb{Q}_l) \to G_l^0(\mathbb{Q}_l)$ is an abelian group of exponent dividing A.
- Let T_l denote a maximal torus in G_l^0 which we assume to be chosen unramified whenever G_l^0 is unramified. Let $T_l^{\mathrm{ad}} = T_l/Z_l$ and $T_l^{\mathrm{der}} = \ker(T_l \to C_l)^0$ and T_l^{SC} the connected preimage of T_l^{ad} in G_l^{SC} and $T_l^H = T_l^{\mathrm{SC}} \times Z_l$. We have natural embeddings $X^*(T_l) \hookrightarrow X^*(T_l^H)$ and $X^*(C_l) \hookrightarrow X^*(Z_l)$, both with cokernel killed by A.
- Let $\Delta \subset X^*(T_l^{\mathrm{ad}}) \subset X^*(T_l)$ denote a basis for the root system of G_l. If $\mu \in X^*(Z_l)$ is a weight of V_l then we can find $m_{\mu,\delta} \in \mathbb{Z}$ for $\delta \in \Delta$ such that $((1/A)\sum_{\delta\in\Delta} m_{\mu,\delta}\delta, \mu)$ is a weight of T_l^H on V_l. Moreover the $m_{\mu,\delta}$ can be bounded independently of l. (The bound will depend only on $1+n[M:\mathbb{Q}]$.)
- If $v|l$ is a prime of F^0 then there is an element $\nu_{\mathrm{HT},v} \in X_*(T_l)$ such that for any algebraic representation ρ of G_l^0 defined over $\mathbb{Q}_l$, the Hodge-Tate numbers of $\rho \circ r_l|_{G_{F^0_v}}$ are the $\langle \mu, \nu_{\mathrm{HT},v}\rangle$, where μ runs over the weights of ρ in $X^*(T_l)$. (See section 1.2 of [Win86].)We see that for $\mu \in X^*(T_l)$ a weight of T_l on V_l that $\langle \mu, \nu_{\mathrm{HT},v}\rangle$ is bounded independently of l (and μ and v). We deduce that if $\mu \in X^*(T_l)$ is a root of G_l then $\langle \mu, \nu_{\mathrm{HT},v}\rangle$ is bounded independently of l (and μ and v). Finally if $\mu \in X^*(Z_l)$ is a weight of Z_l on V_l then $A\mu \in X^*(C_l) \subset X^*(Z_l)$ and thinking of $A\mu \in X^*(C_l) \subset X^*(T_l)$ we see that $\langle A\mu, \nu_{\mathrm{HT},v}\rangle$ is bounded independently of l (and μ and v). (Because $A\mu = A\mu' - \sum_{\delta\in\Delta} m_{\mu,\delta}\delta$ where μ' is a weight of T_l on V_l and the $m_{\mu,\delta}$ are bounded independent of l (and μ).)

- Let $\widetilde{S}$ denote the restriction of scalars from $\mathcal{O}_{F^0}$ to $\mathbb{Z}$ of $\mathbb{G}_m$ and set $S_l = \widetilde{S} \times \mathbb{Q}_l$. Also let $\widetilde{S}^1$ denote the kernel of the norm map $\widetilde{S} \to \mathbb{G}_m$ and set $S_l^1 = \widetilde{S}^1 \times \mathbb{Q}_l$. Note that $X^*(S_l)$ has a natural basis $\mathrm{Hom}\,(F^0, \overline{\mathbb{Q}}_l)$ the set of field homomorphisms from F^0 to $\overline{\mathbb{Q}}_l$. There is a homomorphism $\theta_l : S_l \to C_l$ such that the restriction of θ_l to some open subgroup of $\widetilde{S}(\mathbb{Z}_l)$ equals the composite of r_l with the Artin map. (See sections III.1.2 and III.2.1 of [Ser68].) If r_l is crystalline then this equality holds on all of $\widetilde{S}(\mathbb{Z}_l)$. (Use the same argument as in sections III.1.2 and II.2.1 of [Ser68], but replace the appeal to the Theorem of section III.1.2 of [Ser68] by an appeal to Proposition 6.3 of [CCO09].) Moreover θ_l maps S_l surjectively to C_l (because $\theta_l\widetilde{S}(\mathbb{Z}_l)$ is open in $C_l(\mathbb{Q}_l)$) and hence θ_l also maps S_l^1 surjectively to C_l^1. If $\mu \in X^*(C_l)$ and $\sigma : F^0 \hookrightarrow \overline{\mathbb{Q}}_l$ then the σ Hodge-Tate number of $\mu \circ r_l$ equals the negative of the coefficient of σ when $\mu \circ \theta_l \in X^*(S_l)$ is expressed in terms of the basis $\mathrm{Hom}\,(F^0, \overline{\mathbb{Q}}_l)$. In particular if $\mu \in X^*(Z_l)$ is a weight of Z_l on V_l then $(A\mu) \circ \theta_l = \prod_{\sigma \in \mathrm{Hom}\,(F^0, \overline{\mathbb{Q}}_l)} \sigma^{m_{\mu,\sigma}}$, where $m_{\mu,\sigma}$ is bounded independently of l (and μ), say by a constant B.
- If G_l^0 is unramified let $\widetilde{Z}_l$, $\widetilde{Z}_l^1$, $\widetilde{C}_l$ and $\widetilde{C}_l^1$ denote the tori over $\mathbb{Z}_l$ with generic fibres Z_l, Z_l^1, C_l and C_l^1. Then $\widetilde{Z}_l(\mathbb{Z}_l)$ (resp. $\widetilde{Z}_l^1(\mathbb{Z}_l)$, $\widetilde{C}_l(\mathbb{Z}_l)$, $\widetilde{C}_l^1(\mathbb{Z}_l)$) is the unique maximal compact subgroup of $Z_l(\mathbb{Q}_l)$ (resp. $Z_l^1(\mathbb{Q}_l)$, $C_l(\mathbb{Q}_l)$, $C_l^1(\mathbb{Q}_l)$). In particular the image of Γ_l^0 in $C_l(\mathbb{Q}_l)$ is contained in $\widetilde{C}_l(\mathbb{Z}_l)$. Also the preimage of $\widetilde{C}_l(\mathbb{Z}_l)$ in $Z_l(\mathbb{Q}_l)$ is $\widetilde{Z}_l(\mathbb{Z}_l)$.

According to [Lar95] there is a set $\mathcal{L}$ of rational primes of Dirichlet density 1 with the following properties.

- If $l \in \mathcal{L}$ then l is unramified in M and in F^0, and $\mathcal{R}$ is unramified above l (i.e. no prime of F above l lies in the finite set $S_{\mathcal{R}}$ of bad primes). In particular if $l \in \mathcal{L}$ then r_l is crystalline.
- If $l \in \mathcal{L}$ then $l \geq 4B + 4$.
- If $l \in \mathcal{L}$ then G_l^0 is unramified. (See Section 3.15 of [Lar95].)
- If $l \in \mathcal{L}$ then there is a semi-simple group scheme $\widetilde{G}_l^{\mathrm{SC}}/\mathbb{Z}_l$ with generic fibre G_l^{SC} such that the pre-image in $G_l^{\mathrm{SC}}(\mathbb{Q}_l)$ of the image in $G_l^{\mathrm{ad}}(\mathbb{Q}_l)$ of G_{F^0} equals $\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l)$. (See Theorem 3.17 of [Lar95].)
- If $l \in \mathcal{L}$ then $\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l)$ is a perfect group. (Combine Proposition 2.6 of [Lar95] with the fact that for $l > 3$ the group $\widetilde{G}_l^{\mathrm{SC}}(\mathbb{F}_l)$ is perfect (see for instance section 1.2 of [Lar95]).)
- If $l \in \mathcal{L}$, if $\lambda | l$ is a prime of M, and if s_1 is an irreducible constituent of r_l thought of as a representation of G_l^{SC} then s_1 can be defined over M_λ and extends to a representation $\widetilde{s}_1$ of $\widetilde{G}_l^{\mathrm{SC}}$ over $\mathcal{O}_{M,\lambda}$. Moreover, writing $\overline{s}_1$ for the reduction of $\widetilde{s}_1$, then $\overline{s}_1$ is a standard irreducible representation of $\widetilde{G}_l^{\mathrm{SC}} \times \overline{k(\lambda)}$. (See sections (1.12) and (1.13) of [Lar95], where the term 'standard' is also defined.)

From now on we restrict to $l \in \mathcal{L}$. Set $\widetilde{H}_l = \widetilde{G}_l^{\mathrm{SC}} \times \widetilde{Z}_l$ and $\widetilde{H}_l^1 = \widetilde{G}_l^{\mathrm{SC}} \times \widetilde{Z}_l^1$. We have the equality

$$(\Gamma_l^0 \widetilde{Z}_l(\mathbb{Z}_l)) \cap \mathrm{Im}\,(H_l(\mathbb{Q}_l) \to G_l^0(\mathbb{Q}_l)) = \mathrm{Im}\,(\widetilde{H}_l(\mathbb{Z}_l) \to G_l^0(\mathbb{Q}_l)).$$

[If $g \in \widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l)$ then we can write $g = \gamma z$ in $G_l(\mathbb{Q}_l)$, where $\gamma \in \Gamma_l^0$ and $z \in Z_l(\mathbb{Q}_l)$. We see that the image of z in $C_l(\mathbb{Q}_l)$ lies in $\widetilde{C}_l(\mathbb{Z}_l)$ and hence $z \in \widetilde{Z}_l(\mathbb{Z}_l)$. Thus the image of $\widetilde{H}_l(\mathbb{Z}_l)$ in $G_l^0(\mathbb{Q}_l)$ is contained in $\Gamma_l^0\widetilde{Z}_l(\mathbb{Z}_l)$. Thus the pre-image of $\Gamma_l^0\widetilde{Z}_l(\mathbb{Z}_l)$ in $H_l(\mathbb{Q}_l)$ is a compact subgroup containing $\widetilde{H}_l(\mathbb{Z}_l)$, which is maximal compact in $H_l(\mathbb{Q}_l)$, and so in fact the pre-image equals $\widetilde{H}_l(\mathbb{Z}_l)$. The desired equality follows.] Thus

$$\Gamma_l \cap \mathrm{Im}\,(\widetilde{H}_l(\mathbb{Z}_l) \to G_l^0(\mathbb{Q}_l)) = \ker(\Gamma_l^0 \to (G_l^0(\mathbb{Q}_l)/\mathrm{Im}\,(H_l(\mathbb{Q}_l) \to G_l^0(\mathbb{Q}_l)))).$$

Call this group Γ_l^{00}. We see that Γ_l^{00} is an open normal subgroup of Γ_l^0 and that Γ_l^0/Γ_l^{00} is an abelian group of exponent dividing A. As $\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l)$ is perfect the pre-image of Γ_l^{00} in $H_l(\mathbb{Q}_l)$ is of the form $\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l) \times \Gamma_l^Z$, where Γ_l^Z is an open subgroup of $\widetilde{Z}_l(\mathbb{Z}_l)$. Also let Γ_l^C denote the image of Γ_l^0 in $C_l(\mathbb{Q}_l)$. The cokernel of the map $\Gamma_l^Z \to \Gamma_l^C$ (which equals the cokernel of the map $\Gamma_l^{00} \to \Gamma_l^C$) is killed by A. Set $\Gamma_l^{Z,1} = \Gamma_l^Z \cap Z_l^1(\mathbb{Q}_l)$ and $\Gamma_l^{C,1} = \Gamma_l^C \cap C^1(\mathbb{Q}_l)$, so that $\Gamma_l^{Z,1}$ is the pre-image of $\Gamma_l^{C,1}$ under $\Gamma_l^Z \to \Gamma_l^C$. Again the cokernel of $\Gamma_l^{Z,1} \to \Gamma_l^{C,1}$ is killed by A.

As G_l acts by conjugation on G_l^0 it also acts on Z_l, $\widetilde{Z}_l$, C_l, $\widetilde{C}_l$, G_l^{ad}, G_l^{der}, G_l^{SC} and H_l. As ϵ_l is a character of G_l the conjugation action of G_l also preserves Z_l^1, $\widetilde{Z}_l^1$, C_l^1 and $\widetilde{C}_l^1$. In particular Γ_l^{00} is normal in Γ_l. If $\gamma \in \Gamma_l$ then we see that $({}^\gamma\widetilde{H}_l)(\mathbb{Z}_l) = \widetilde{H}_l(\mathbb{Z}_l)$ (both being the preimage of $\widetilde{Z}_l(\mathbb{Z}_l)\Gamma_l^0$ in $H_l(\mathbb{Q}_l)$), so that ${}^\gamma\widetilde{H}_l = \widetilde{H}_l$. [Being a hyperspecial maximal compact $\widetilde{H}_l(\mathbb{Z}_l)$ fixes a unique point x in the reduced building $\mathcal{B}$ of H_l. Thus $\gamma x = x$. Thus $({}^\gamma\widetilde{H}_l)(\mathbb{Z}_{l^r}) = \widetilde{H}_l(\mathbb{Z}_{l^r})$ for all $r \geq 1$ (as they are both equal to the stabilizer of x in $H_l(\mathbb{Q}_{l^r})$. This implies that ${}^\gamma\widetilde{H}_l = \widetilde{H}_l$, because the affine ring of each is the space of functions in $\mathbb{Q}_l[H_l]$ which take integer values on $\widetilde{H}_l(\mathbb{Z}_{l^r}) = ({}^\gamma\widetilde{H}_l)(\mathbb{Z}_{l^r})$ for all r. We thank Jiu-Kang Yu for supplying us with this argument.] We conclude that $\widetilde{H}_l$ and $\widetilde{H}_l^1$ also have actions of Γ_l compatible with the actions on H_l and H_l^1.

Now let s_1 and s_2 denote two irreducible sub-representations of $r_\lambda|_{G_{F^0}} \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l$. This is the same thing as being irreducible G_l^0-submodules of $V_l \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l$, or even H_l-submodules. We see that s_1 and s_2 can be defined over M_λ. Suppose that $\overline{s}_1|_{\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l)\times\Gamma_l^{Z,1}} \cong \overline{s}_2|_{\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l)\times\Gamma_l^{Z,1}}$, then $s_1|_{H_l} \cong (s_2 \otimes \epsilon_l^a)|_{H_l}$ for some $a \in \mathbb{Z}$. [From $\overline{s}_1|_{\widetilde{G}_l^{\mathrm{SC}}(\mathbb{F}_l)} \cong \overline{s}_2|_{\widetilde{G}_l^{\mathrm{SC}}(\mathbb{F}_l)}$ we deduce that $\overline{s}_1 \cong \overline{s}_2$ (see for instance the first paragraph of section (1.13) of [Lar95]) and hence that $s_1|_{G_l^{\mathrm{SC}}} \cong s_2|_{G_l^{\mathrm{SC}}}$ (see for instance the last paragraph of section (1.12) of [Lar95] and recall that the $\overline{s}_i$ are standard). Write $\mu_i = s_i|_{Z_l} \in X^*(Z_l)$ and let $\overline{\mu}_i$ denote its reduction modulo l. Then $\mu_i^A \circ \theta_l = \prod_{\sigma \in \mathrm{Hom}\,(F^0,\overline{\mathbb{Q}}_l)} \sigma^{m_{i,\sigma}}$ where $m_{i,\sigma} \in \mathbb{Z}$ and $|m_{i,\sigma}| < B$. As $\overline{\mu}_1|_{\Gamma_l^{Z,1}} = \overline{\mu}_2|_{\Gamma_l^{Z,1}}$ we have $\overline{\mu}_1^A|_{\Gamma_l^{C,1}} = \overline{\mu}_2^A|_{\Gamma_l^{C,1}}$ and thus for some b with $(1-l)/2 < b < (l-1)/2$ we have $\overline{\mu}_1^A|_{\Gamma_l^C} = (\epsilon_l^b\overline{\mu}_2^A)|_{\Gamma_l^C}$. As μ_1^A and μ_2^A are crystalline (as characters of G_{F^0}) we deduce that

$$\prod_{\sigma \in \mathrm{Hom}\,(F_0,\mathbb{Q}_l^{\mathrm{nr}})} \overline{\sigma}^{m_{1,\sigma}-m_{2,\sigma}} = \prod_{\sigma \in \mathrm{Hom}\,(F_0,\mathbb{Q}_l^{\mathrm{nr}})} \overline{\sigma}^b$$

on $(\mathcal{O}_{F^0}/l)^\times$. If $v|l$ is a prime of F^0, choose an embedding $\sigma_v : F^0 \hookrightarrow \mathbb{Q}_l^{\mathrm{nr}}$ above v. Then the embeddings of F_v^0 into $\mathbb{Q}_l^{\mathrm{nr}}$ are $\mathrm{Frob}_l^i \circ \sigma_v$ for $i = 0, \ldots, f_v - 1$ with $f_v = [k(v) : \mathbb{F}_l]$. Then

$$\sum_{i=0}^{f_v-1} (m_{1,\mathrm{Frob}_l^i\circ\sigma_v} - m_{2,\mathrm{Frob}_l^i\circ\sigma_v})l^i \equiv b(l^{f_v}-1)/(l-1) \bmod (l^{f_v}-1).$$

As $(1-l)/2 < m_{1,\sigma} - m_{2,\sigma} < (l-1)/2$ for all σ, both sides lie in the range $((1-l^{f_v})/2, (l^{f_v}-1)/2)$ and so

$$\sum_{i=0}^{f_v-1} (m_{1,\mathrm{Frob}_l^i \circ \sigma_v} - m_{2,\mathrm{Frob}_l^i \circ \sigma_v}) l^i = b(l^{f_v}-1)/(l-1).$$

Then we see that $m_{1,\mathrm{Frob}_l^0 \circ \sigma_v} - m_{2,\mathrm{Frob}_l^0 \circ \sigma_v} \equiv b \bmod l$ and again using the bounds on both sides we conclude that $m_{1,\mathrm{Frob}_l^0 \circ \sigma_v} - m_{2,\mathrm{Frob}_l^0 \circ \sigma_v} = b$. Subtracting these terms, dividing by l and arguing recursively we see that

$$m_{1,\mathrm{Frob}_l^i \circ \sigma_v} - m_{2,\mathrm{Frob}_l^i \circ \sigma_v} = b$$

for all i. Thus

$$\mu_1^A \circ \theta_l = (\mu_2^A \circ \theta_l) \otimes \epsilon_l^b$$

so that $\mu_1^A = \mu_2^A \epsilon_l^b$ on C_l and $\mu_1 = \mu_2$ on Z_l^1. The claim follows.]

We can think of s in the statement of the Proposition as a representation of $H_l \rtimes \Gamma_l$. (The restriction of this representation to $\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l) \times \Gamma_l^Z \subset H_l(\mathbb{Q}_l)$ equals the composite of projection to Γ_l^{00} with the restriction of the representation to $\Gamma_l^{00} \subset \Gamma_l$.) As in section 1.12 of [Lar95] we see that there is a $\mathcal{O}_{\mathbb{Q}_l^{\mathrm{nr}}}$-lattice Λ in $s \otimes_{M_\lambda} \mathbb{Q}_l^{\mathrm{nr}}$ which is invariant under $\widetilde{H}_l(\mathcal{O}_{\mathbb{Q}_l^{\mathrm{nr}}})$ (and hence also under Γ_l^{00}). Replacing Λ by the sum of its Γ_l/Γ_l^{00}-translates we may assume that Λ is also invariant under Γ_l. Again as in section 1.12 of [Lar95] we see that $s|_{H_l \rtimes \Gamma_l}$ extends to a homomorphism $\widetilde{s} : (\widetilde{H}_l \times_{\mathbb{Z}_l} \mathcal{O}_{M,\lambda}) \rtimes \Gamma_l \to GL_d \times \mathcal{O}_{M,\lambda}$. (Here $d = \dim s$.)

Now let s_0 denote an irreducible Γ_l^0-submodule of s and let Γ_l^1 denote the set of γ in Γ_l with $s_0^\gamma \cong s_0$, or what comes to the same thing (by regularity) $s_0^\gamma = s_0 \subset s$. Then s_0 extends to a representation of Γ_l^1 and $s \cong \mathrm{Ind}_{\Gamma_l^1}^{\Gamma_l} s_0$. We may also think of s_0 as a representation of G_l^0, or of H_l, and Γ_l^1 is the set of $\gamma \in \Gamma_l$ with $s_0^\gamma \cong s_0$ as a representations of H_l. We obtain an integral model for s_0 by taking $s_0 \cap \mathcal{O}_{M_\lambda}^d$. As l is unramified in F^0 we see that $[\Gamma_l : \Gamma_l^1] = [\ker \overline{\epsilon}_l|_{\Gamma_l} : \ker \overline{\epsilon}_l|_{\Gamma_l^1}]$ and so

$$\overline{s}|_{G_{F(\zeta_l)}} \cong \mathrm{Ind}_{\ker \overline{\epsilon}_l|_{\Gamma_l^1}}^{\ker \overline{\epsilon}_l|_{\Gamma_l}} \overline{s}_0.$$

We have seen that $\overline{s}_0|_{\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l) \times \Gamma_l^{Z,1}}$ is irreducible. If $\gamma \in \Gamma_l$ and $\overline{s}_0|_{\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l) \times \Gamma_l^{Z,1}} \cong \overline{s}_0|^\gamma_{\widetilde{G}_l^{\mathrm{SC}}(\mathbb{Z}_l) \times \Gamma_l^{Z,1}}$ then $s_0^\gamma \cong s_0 \epsilon_l^a$ as representations of H_l for some a. However as γ has finite order in Γ_l/Γ_l^0 we conclude that we must have $a = 0$ and so $\gamma \in \Gamma_l^1$. Thus

$$\overline{s}|_{G_{F(\zeta_l)}} \cong \mathrm{Ind}_{\ker \overline{\epsilon}_l|_{\Gamma_l^1}}^{\ker \overline{\epsilon}_l|_{\Gamma_l}} \overline{s}_0$$

is irreducible as desired. $\square$

Lemma 5.2.3. *Suppose that F is a CM (or totally real) field; that $\mathcal{R}$ is a pure, extremely regular, weakly compatible system of l-adic representations of G_F defined over M; and that $\mathcal{M}$ is a weakly compatible system of characters of G_{F^+} defined over M such that $(\mathcal{R}, \mathcal{M})$ is essentially conjugate self-dual. If F'/F is a finite extension and if s is a sub-representation of $r_\lambda|_{G_{F'}}$ for some prime λ of M, then there is a CM field F'' with $F \subset F'' \subset F'$ such that s is invariant by $G_{F''}$. Moreover, the pair (s, μ_λ) is essentially conjugate self-dual, and is totally odd, essentially conjugate self-dual if (r_λ, μ_λ) is.*

Proof. Let F_1 denote the normal closure of F'/F^+. Let $\tau : F \hookrightarrow \overline{M}$ be an embedding with the property that if H and H' are different subsets of H_τ of the same cardinality then $\sum_{h\in H} h \neq \sum_{h\in H'} h$. Choose an embedding $\tau_1 : F_1 \hookrightarrow \overline{M}$ extending τ. Note that $\mathcal{R}$ is pure of some weight w.

If s_1 and s_2 are two G_{F_1}-submodules of some $r_{\lambda'}$ of the same dimension we see that $s_1 = s_2$ if and only if $\mathrm{HT}_{\tau_1}(s_1) = \mathrm{HT}_{\tau_1}(s_2)$ if and only if $\mathrm{HT}_{\tau_1}(\det s_1) = \mathrm{HT}_{\tau_1}(\det s_2)$. (The first equivalence by regularity and the second by extreme regularity. Note that in particular any irreducible submodule of $r_{\lambda'}|_{G_{F_1}}$ has multiplicity 1.)

For $\sigma \in \mathrm{Gal}\,(F_1/F^+)$ write $\mathrm{HT}_{\tau_1}((\det s)^\sigma) = \mathrm{HT}_{\tau_1\circ\sigma^{-1}}(\det s) = \{h_\sigma\}$. As $\det s$ is de Rham and pure of weight $w \dim s$, we deduce that $h_\sigma + h_{\sigma c} = w\dim s$ for all $\sigma \in \mathrm{Gal}\,(F_1/F^+)$ and all complex conjugations $c \in \mathrm{Gal}\,(F_1/F^+)$. Thus if $c, c' \in \mathrm{Gal}\,(F_1/F^+)$ are complex conjugations then $h_{\sigma cc'} = h_\sigma$ and so $s^{\sigma cc'} = s^\sigma$. Let $H \subset \mathrm{Gal}\,(F_1/F^+)$ be the normal subgroup generated by all elements cc' with $c, c' \in \mathrm{Gal}\,(F_1/F^+)$ complex conjugations. Then $F'' = (F_1)^{H\mathrm{Gal}\,(F_1/F')}$ is the maximal CM sub-field of F'. Moreover if $\sigma \in H$ then $s^\sigma = s$ and so s extends to a representation of $G_{F''}$. Moreover if $c \in \mathrm{Gal}\,(F_1/F^+)$ is a complex conjugation then $\mathrm{HT}_{\tau_1}(\det(\mu_\lambda(s^\vee)^c)) = \mathrm{HT}_{\tau_1 c}(\det(\mu_\lambda s^\vee)) = \{w\dim s - h_c\} = \{h_1\}$. As $\mu_\lambda(s^\vee)^c$ is also a constituent of r_λ, we see that $s^c \cong \mu_\lambda s^\vee$ as representations of $G_{F''}$. Let v be an infinite place of F and $\langle\ ,\ \rangle_v$ a pairing on $\overline{M}_\lambda^n$ as in the definition of essential conjugate self-duality for the pair (r_λ, μ_λ). Let $A_{\lambda,v} \in GL_n(\overline{M}_\lambda)$ be the symmetric or antisymmetric matrix corresponding to the pairing $\langle\ ,\ \rangle_v$. Then the isomorphism $s^c \cong \mu_\lambda s^\vee$ arises from the restriction of the matrix $A_{\lambda,v}$ to s. The last sentence of the lemma follows. $\square$

5.3. **Potential automorphy for weakly compatible systems.**

In this section we prove a potential automorphy theorem for weakly compatible systems of l-adic representations of the absolute Galois group of a CM or totally real field.

Theorem 5.3.1. *Suppose that F is a CM (or totally real) field and that $\mathcal{R}$ is an irreducible, totally odd, essentially conjugate self-dual, regular, weakly compatible system of l-adic representations of G_F. Then there is a finite, CM (or totally real), Galois extension F'/F such that $\mathcal{R}|_{G_{F'}}$ is automorphic.*

Proof. The totally real case follows easily from the CM case and Lemma 1.5 of [BLGHT09], so we treat only the CM case. By Propositions 2.1.2 and 5.2.2 we can find a set $\mathcal{L}$ of primes λ of M with Dirichlet density 1 so that for $\lambda \in \mathcal{L}$ the image $\overline{r}_\lambda(G_{F(\zeta_l)}) \subset GL_n(\overline{\mathbb{F}}_l)$ is adequate. Removing finitely many primes from $\mathcal{L}$ we may also assume that for $\lambda \in \mathcal{L}$ we have that l is unramified in F and r_λ is crystalline at all primes of F above l with Hodge-Tate numbers in a range of the form $[a, a+l-2]$. Our theorem now follows by applying Theorem 4.5.1 to r_λ for any $\lambda \in \mathcal{L}$. $\square$

Corollary 5.3.2. *Keep the assumptions of the theorem.*

1. *If $\imath : M \hookrightarrow \mathbb{C}$, then $L^S(\imath\mathcal{R}, s)$ converges (uniformly absolutely on compact subsets) on some right half plane and has meromorphic continuation to the whole complex plane.*

(2) *The compatible system $\mathcal{R}$ is strictly pure. Moreover*

$$\Lambda(\imath\mathcal{R}, s) = \epsilon(\imath\mathcal{R}, s)\Lambda(\imath\mathcal{R}^\vee, 1-s).$$

(3) *If F is totally real, n is odd, and $v|\infty$ then $\operatorname{tr} r_\lambda(c_v) = \pm 1$ and is independent of λ.*

Proof. The strict purity follows from the theorem, the results of [Car10] and the usual Brauer's theorem argument as in the last paragraph of the proof of Theorem 5.4.1. The convergence and meromorphic continuation and functional equation of the L-function follow from the theorem and a Brauer's theorem argument as in Theorem 4.2 of [HSBT10]. The last part generalizes an observation of F. Calegari [Cal09]. The theorem reduces the question to the automorphic case where it is the main result of [Tay10]. $\square$

As one example of the above theorem we state the following result.

Corollary 5.3.3. *Suppose that $\mathcal{K}$ is a finite set of positive integers with the property that the $2^{\#\mathcal{K}}$ partial sums of elements of $\mathcal{K}$ are all distinct. For each $k \in \mathcal{K}$ let f_k be an elliptic modular newform of weight $k+1$ without complex multiplication and let π_k be the corresponding automorphic representation of $GL_2(\mathbb{A})$. Then there is a totally real Galois extension $F/\mathbb{Q}$ and a RAESDC automorphic representation Π of $GL_{2^{\#\mathcal{K}}}(\mathbb{A}_F)$ such that for all but finitely many primes v of F we have*

$$\operatorname{rec}(\Pi_v |\det|_v^{(1-2^{\#\mathcal{K}})/2}) = \left.\left(\bigotimes_{k\in\mathcal{K}} \operatorname{rec}(\pi_{k,v|_\mathbb{Q}} |\det|_{v|_\mathbb{Q}}^{-1/2})\right)\right|_{W_{F_v}}.$$

In particular the 'multiple product' L-function $L(\times_{k\in\mathcal{K}}\pi_k, s)$ has meromorphic continuation to the whole complex plane.

Proof. Let M denote the compositum of the fields of coefficients of the f_k's. Let λ be any prime of M and let $r_{k,\lambda} : G_\mathbb{Q} \to GL_2(\overline{M}_\lambda)$ be the λ-adic representation associated to f_k. Because f_k is not CM we know that $r_{k,\lambda}$ has Zariski dense image. We will apply Theorem 5.3.1 to the weakly compatible system

$$\bigotimes_{\mathcal{K}} r_{k,\lambda}.$$

The only assumption, which is perhaps not clear, is that this system is irreducible. So it only remains to check this property.

Now let H denote the Zariski closure of $(\prod_\mathcal{K} r_{k,\lambda})(G_\mathbb{Q})$ in $GL_2(\overline{M}_\lambda)^\mathcal{K}$ and let $\overline{H}$ denote its image in $PGL_2(\overline{M}_\lambda)^\mathcal{K}$. Note that the projection of $\overline{H}$ to each factor is surjective. As PGL_2 is a simple algebraic group and all its automorphisms are inner, $\overline{H}$ must be of the form $PGL_2(\overline{M}_\lambda)^\mathcal{I}$ for some set $\mathcal{I}$. Moreover we can decompose $\mathcal{K} = \coprod_{i\in\mathcal{I}} \mathcal{K}_i$ and the mapping $\overline{H} \to PGL_2(\overline{M}_\lambda)^\mathcal{K}$ is conjugate to the mapping which sends the i^{th} factor of $PGL_2(\overline{M}_\lambda)^\mathcal{I}$ diagonally into $\prod_{i\in\mathcal{K}_i} PGL_2(\overline{M}_\lambda)$. If for some i one had $\#\mathcal{K}_i > 1$ then we would have $r_{k,\lambda} = r_{k',\lambda}\otimes\chi$ for some $k \neq k'$ in $\mathcal{K}_i$ and some character χ. We can conclude that χ is de Rham and then looking at Hodge-Tate numbers gives a contradiction. Thus we must have $\overline{H} = PGL_2(\overline{M}_\lambda)^\mathcal{K}$ and $H \supset SL_2(\overline{M}_\lambda)^\mathcal{K}$. As the tensor product representation of $SL_2(\overline{M}_\lambda)^\mathcal{K}$ on $\overline{M}_\lambda^{2^{\#\mathcal{K}}}$ is irreducible we conclude that $\bigotimes_\mathcal{K} r_{k,\lambda}$ is also irreducible, as desired. $\square$

The next proposition will be useful in the next section. Its proof is the same as the proof of Theorem 5.3.1, except that we must also appeal to Lemma 5.2.3 to ensure the needed conjugate self-duality.

Proposition 5.3.4. *Suppose that F is a CM (or totally real) field; that $\mathcal{R}$ is a pure, extremely regular, weakly compatible system of l-adic representations of G_F; and that $\mathcal{M}$ is a weakly compatible system of characters of G_{F^+} such that $(\mathcal{R},\mathcal{M})$ is totally odd, essentially conjugate self-dual. Write $r_\lambda = r_{\lambda,1}\oplus\cdots\oplus r_{\lambda,j_\lambda}$ with each $r_{\lambda,\alpha}$ irreducible.Then there is a set of rational primes $\mathcal{L}$ of Dirichlet density 1 such that if λ is a prime of M lying above $l\in\mathcal{L}$, then there is a finite, CM (or totally real), Galois extension F'/F and RAECSDC (or RAESDC) automorphic representations (π_α,χ_α) of $GL_{n_\alpha}(\mathbb{A}_{F'})$ where $n_\alpha=\dim r_{\lambda,\alpha}$, such that, for each $\alpha=1,\ldots,j_\lambda$ the restriction $r_{\lambda,\alpha}|_{G_{F'}}$ is irreducible and $(r_{l,\imath}(\pi_\alpha),\epsilon_l^{1-n_\alpha}r_{l,\imath}(\chi_\alpha))\cong(r_{\lambda,\alpha}|_{G_{F'}},\mu_\lambda|_{G_{F'}})$ (for some $\imath:\overline{M}_\lambda\overset{\sim}{\to}\mathbb{C}$).*

Proof. The totally real case follows easily from the CM case and Lemma 1.5 of [BLGHT09], so we treat only the CM case.

By Lemma 5.2.3 we see that for all λ and α the pair $(r_{\lambda,\alpha},\mu_\lambda)$ is totally odd, essentially conjugate self-dual. By Propositions 2.1.2 and 5.2.2 we can find a set $\mathcal{L}$ of rational primes with Dirichlet density 1 so that for $\lambda|l\in\mathcal{L}$ the image $\overline{r}_{\lambda,\alpha}(G_{F(\zeta_l)})\subset GL_{n_\alpha}(\overline{\mathbb{F}}_l)$ is adequate. Removing finitely many primes from $\mathcal{L}$ we may also assume that for $\lambda|l\in\mathcal{L}$ we also have that l is unramified in F and r_λ is crystalline at all primes of F above l with Hodge-Tate numbers in a range of the form $[a,a+l-2]$. Our Theorem now follows by applying Theorem 4.5.1 to $\{r_{\lambda,\alpha}\}_\alpha$ for any $\lambda|l\in\mathcal{L}$ with $F^{(\mathrm{avoid})}$ equal to the compositum of the $\overline{F}^{\ker\overline{r}_{\lambda,\alpha}}$ for $\alpha=1,\ldots,j_\lambda$. Then $\overline{r}_{\lambda,\alpha}|_{G_{F'}}$ will be irreducible for $\alpha=1,\ldots,j_\lambda$, and so $r_{\lambda,\alpha}|_{G_{F'}}$ will also be irreducible. □

5.4. **Irreducibilty results.**

We will first recall some basic group theory. If F is a number field and l is a rational prime we will let $\mathrm{GG}_{F,l}$ denote the category of semi-simple, continuous representations of G_F on finite dimensional $\overline{\mathbb{Q}}_l$-vector spaces which ramify at only finitely many primes. If U, V and W are objects of $\mathrm{GG}_{F,l}$ with $U\oplus W\cong V\oplus W$ then $U\cong V$ (because they have the same traces). We will let $\mathrm{Rep}_{F,l}$ denote the Grothendieck group of $\mathrm{GG}_{F,l}$. If V is an object of $\mathrm{GG}_{F,l}$ we will denote by $[V]$ its class in $\mathrm{Rep}_{F,l}$. We have the following functorialities.

1. The rule $[U][V]=[U\otimes V]$ makes $\mathrm{Rep}_{F,l}$ a commutative ring with 1.
2. If $\sigma\in G_F$ then there is a ring homomorphism $\mathrm{tr}_\sigma:\mathrm{Rep}_{F,l}\to\overline{\mathbb{Q}}_l$ defined by $\mathrm{tr}_\sigma[V]=\mathrm{tr}\,\sigma|_V$. If $A\in\mathrm{Rep}_{F,l}$ then the function $\sigma\mapsto\mathrm{tr}_\sigma A$ is a continuous class function $G_F\to\overline{\mathbb{Q}}_l$. If $A,B\in\mathrm{Rep}_{F,l}$ and $\mathrm{tr}_\sigma A=\mathrm{tr}_\sigma B$ for all $\sigma\in G_F$ (or even for a dense set of σ) then $A=B$.
3. We will write dim for tr_1. Then in fact $\dim:\mathrm{Rep}_{F,l}\to\mathbb{Z}$ and $\dim[V]=\dim_{\overline{\mathbb{Q}}_l}V$.
4. There is a perfect symmetric $\mathbb{Z}$-valued pairing $(\ ,\ )_{F,l}$ on $\mathrm{Rep}_{F,l}$ defined by
$$([U],[V])_{F,l}=\dim_{\overline{\mathbb{Q}}_l}\mathrm{Hom}_{G_F}(U,V).$$
If $A=\sum_i n_i[V_i]$ with the V_i irreducible and distinct then $(A,A)=\sum_i n_i^2$. In particular if $A\in\mathrm{Rep}_{F,l}$ and $\dim A\geq 0$ and $(A,A)=1$ then $A=[V]$ for some irreducible object V of $\mathrm{GG}_{F,l}$.

(5) Suppose that G_1 and G_2 are algebraic groups over $\overline{\mathbb{Q}}_l$ and that $\theta : G_F \to G_1(\overline{\mathbb{Q}}_l) \times G_2(\overline{\mathbb{Q}}_l)$ is a continuous homomorphism with Zariski dense image. Suppose also that ρ_i and ρ_i' are semi-simple algebraic representations of G_i over $\overline{\mathbb{Q}}_l$. Then

$$([(\rho_1 \otimes \rho_2) \circ \theta], [(\rho_1' \otimes \rho_2') \circ \theta])_{F,l} = ([\rho_1 \circ \theta], [\rho_1' \circ \theta])_{F,l}([\rho_2 \circ \theta], [\rho_2' \circ \theta])_{F,l}.$$

(6) If $\sigma \in G_{\mathbb{Q}}$ then there is a ring isomorphism conj_σ from $\mathrm{Rep}_{F,l}$ to $\mathrm{Rep}_{\sigma^{-1}F,l}$ such that $\mathrm{conj}_\sigma[V]$ equals the class of the representation of $G_{\sigma^{-1}F}$ on V, under which τ acts by $\sigma\tau\sigma^{-1}$. It preserves dimension, and takes $(\ ,\ )_{F,l}$ to $(\ ,\ )_{\sigma^{-1}F,l}$. We have $\mathrm{tr}\,_\tau \mathrm{conj}_\sigma A = \mathrm{tr}\,_{\sigma\tau\sigma^{-1}} A$. Also if $\sigma \in G_F$ then conj_σ is the identity on $\mathrm{Rep}_{F,l}$.

(7) If F'/F is a finite extension then the formula $\mathrm{res}_{F'/F}[V] = [V|_{G_{F'}}]$ defines a ring homomorphism $\mathrm{res}_{F'/F} : \mathrm{Rep}_{F,l} \to \mathrm{Rep}_{F',l}$. Note that $\mathrm{tr}\,_\sigma \mathrm{res}_{F'/F} A = \mathrm{tr}\,_\sigma A$ (so in particular $\dim \mathrm{res}_{F'/F} A = \dim A$) and that $\mathrm{conj}_\sigma \circ \mathrm{res}_{F'/F} = \mathrm{res}_{\sigma^{-1}F'/\sigma^{-1}F} \circ \mathrm{conj}_\sigma$.

(8) If F'/F is a finite extension there is a $\mathbb{Z}$-linear map $\mathrm{ind}_{F'/F} : \mathrm{Rep}_{F',l} \to \mathrm{Rep}_{F,l}$ defined by $\mathrm{ind}_{F'/F}[V] = [\mathrm{Ind}\,_{G_{F'}}^{G_F} V]$. Note the following.
   (a) $\mathrm{tr}\,_\sigma \mathrm{ind}_{F'/F} A = \sum_{\tau \in G_F/G_{F'}:\ \tau\sigma\tau^{-1} \in G_{F'}} \mathrm{tr}\,_{\tau\sigma\tau^{-1}} A$.
   (b) $\dim \mathrm{ind}_{F'/F} A = [F' : F] \dim A$.
   (c) $\mathrm{ind}_{F'/F}(A(\mathrm{res}_{F'/F} B)) = (\mathrm{ind}_{F'/F} A)B$.
   (d) $(\mathrm{ind}_{F'/F} A, B)_{F,l} = (A, \mathrm{res}_{F'/F} B)_{F',l}$. (By Frobenius reciprocity.)
   (e) If F''/F is another finite extension then

$$\mathrm{res}_{F''/F} \circ \mathrm{ind}_{F'/F} = \sum_{[\sigma] \in G_{F'} \backslash G_F / G_{F''}} \mathrm{ind}_{(\sigma^{-1}F').F''/F''} \circ \mathrm{conj}_\sigma \circ \mathrm{res}_{F'.(\sigma F'')/F'}.$$

   (By Mackey's formula.)
   (f) If F'/F is a finite Galois extension then there are intermediate fields $F'/F_i'/F$ with F'/F_i' soluble; characters $\psi_i : \mathrm{Gal}\,(F'/F_i') \to \mathbb{C}^\times$; and integers n_i such that

$$1 = \sum_i n_i \mathrm{ind}_{F_i'/F}[\psi_i]$$

   in the Grothendieck group of finite dimensional representation of the finite group $\mathrm{Gal}\,(F'/F)$ over $\mathbb{C}$. (This is just Brauer's theorem for $\mathrm{Gal}\,(F'/F)$.) If $\imath : \overline{\mathbb{Q}}_l^\times \xrightarrow{\sim} \mathbb{C}$ then applying $\imath^{-1}$ and multiplying by any $A \in \mathrm{Rep}_{F,l}$ we conclude that

$$A = \sum_i n_i \mathrm{ind}_{F_i'/F}([\imath^{-1}\psi_i] \mathrm{res}_{F_i'/F} A).$$

   Writing

$$G_F = \coprod_k G_{F_i'} \sigma_{ijk} G_{F_j'}$$

   we see further that if

$$A = \sum_i n_i \mathrm{ind}_{F_i'/F}([\imath^{-1}\psi_i] B_i)$$

   then

$$(A, A)_{F,l} = \sum_{i,j,k} n_i n_j \\ ((\mathrm{conj}_{\sigma_{ijk}} \circ \mathrm{res}_{F_i'.(\sigma_{ijk}F_j')/F_i'})([\imath^{-1}\psi_i] B_i), \mathrm{res}_{(\sigma_{ijk}^{-1}F_i').F_j'/F_j'}([\imath^{-1}\psi_j] B_j))_{(\sigma_{ijk}^{-1}F_i').F_j',l}.$$

(9) If S is a finite set of primes of F including all those above l we will say that $A \in \mathrm{Rep}_{F,l}$ is unramified outside S if we can write $A = \sum_i n_i[V_i]$ with each V_i unramified outside S. In this case, we can define, for each $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$,
$$L^S(\imath A, s) = \prod_i L^S(\imath V_i, s)^{n_i}$$
at least as a formal Euler product, which will converge in some right half plane if each V_i is pure (see the paragraph after Proposition 1.6 of [TY07] for the definition of 'pure'). This definition is independent of the choices and we have
$$L^S(\imath(A+B), s) = L^S(\imath A, s)L^S(\imath B, s)$$
and
$$L^S(\imath \mathrm{ind}_{F'/F} A, s) = L^{S'}(\imath A, s),$$
where S' denotes the set of primes of F' above S.

Our first result is not really an irreducibility result, but it uses similar methods so we include it here. It is a generalization of results of Dieulefait [Die02] in dimension 2. The key ingredient is Theorem 4.5.1.

Theorem 5.4.1. *Suppose that F is a CM (or totally real) field, that n is a positive integer and that $l \geq 2(n+1)$ is a rational prime such that $\zeta_l \notin F$. Suppose also that $\mu : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ is a continuous character and that $r : G_F \to GL_n(\overline{\mathbb{Q}}_l)$ is a continuous representation. Suppose moreover that the following conditions are satisfied.*

(1) **(Being unramified almost everywhere)** *r is unramified at all but finitely many primes.*
(2) **(Odd essentially self-duality)** *(r, μ) is totally odd, essentially conjugate self-dual.*
(3) **(Potential diagonalizability and regularity)** *r is potentially diagonalizable (and hence potentially crystalline) at each prime v of F above l and for each $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ the multiset $\mathrm{HT}_\tau(r)$ contains n distinct elements.*
(4) **(Irreducibility)** *$\bar{r}|_{G_{F(\zeta_l)}}$ is irreducible.*

Then r is part of a strictly pure compatible system of l-adic representations of G_F.

Proof. Let G denote the Zariski closure of the image of r, let G^0 denote the connected component of G and let $F^0 = \overline{F}^{r^{-1}G^0(\overline{\mathbb{Q}}_l)}$. By Theorem 4.5.1 (or Corollary 4.5.2) we can find a finite Galois CM (or totally real) extension F'/F, which is linearly disjoint from $F^0\overline{F}^{\ker \bar{r}}$ over F, an isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, and a RAECSDC (or RAESDC) automorphic representation (π, χ) of $GL_n(\mathbb{A}_{F'})$ such that $r_{l,\imath}(\pi) \cong r|_{G_{F'}}$. Suppose that $F' \supset F'' \supset F$ with F'/F'' soluble, then by Lemma 1.4 of [BLGHT09] there is a RAECSDC (or RAESDC) automorphic representation $(\pi^{(F'')}, \chi^{(F'')})$ of $GL_n(\mathbb{A}_{F''})$ such that $r_{l,\imath}(\pi^{(F'')}) \cong r|_{G_{F''}}$.

Let l' be a prime and let $\imath' : \overline{\mathbb{Q}}_{l'} \xrightarrow{\sim} \mathbb{C}$. Note that if $\sigma \in G_F$ and if $F' \supset F'' \supset F''' \supset F$ with F'/F''' soluble then $r_{l',\imath'}(\pi^{(F''')})|_{G_{F''}} \cong r_{l',\imath'}(\pi^{(F'')})$ and $r_{l',\imath'}(\pi^{(F'')})^\sigma \cong r_{l',\imath'}(\pi^{(\sigma^{-1}F'')})$. Let G' (resp. $G^{(F'')} \supset G'$) denote the Zariski closure of $r_{l',\imath'}(\pi)$ (resp. $r_{l',\imath'}(\pi^{(F'')})$) and let $(G')^0$ (resp. $(G^{(F'')})^0$) denote the connected component of G' (resp. $G^{(F'')}$). It follows from Lemma 5.2.1 that

$\mathrm{Gal}\,(F^0F''/F'') \stackrel{\sim}{\to} G^{(F'')}/(G^{(F'')})^0$. We deduce that $G^{(F'')} = G'$ and that the natural map

$$G_{F''} \longrightarrow G'(\overline{\mathbb{Q}}_l) \times \mathrm{Gal}\,(F'/F'')$$

has Zariski dense image. If we decompose $r_{l',\imath'}(\pi)$ into irreducibles as

$$r_{l',\imath'}(\pi) = r_{l',\imath'}(\pi)_1 \oplus \cdots \oplus r_{l',\imath'}(\pi)_t$$

then this induces a unique decomposition

$$r_{l',\imath'}(\pi^{(F'')}) = r_{l',\imath'}(\pi^{(F'')})_1 \oplus \cdots \oplus r_{l',\imath'}(\pi^{(F'')})_t$$

with $r_{l',\imath'}(\pi^{(F'')})_\alpha|_{G_{F'}} \cong r_{l',\imath'}(\pi)_\alpha$. (Note that by regularity the $r_{l',\imath'}(\pi)_\alpha$ are pairwise non-isomorphic, as they will have different Hodge-Tate numbers.) We deduce that if $\sigma \in G_F$ and if $F' \supset F'' \supset F''' \supset F$ with F'/F''' soluble then $r_{l',\imath'}(\pi^{(F''')})_\alpha|_{G_{F''}} \cong r_{l',\imath'}(\pi^{(F'')})_\alpha$ and $r_{l',\imath'}(\pi^{(F'')})^\sigma_\alpha \cong r_{l',\imath'}(\pi^{(\sigma^{-1}F'')})_\alpha$. Moreover if ρ_1 and ρ_2 are representations of $\mathrm{Gal}\,(F'/F'')$ over $\overline{\mathbb{Q}}_{l'}$ then

$$([r_{l'\imath'}(\pi^{(F'')})_\alpha][\rho_1], [r_{l'\imath'}(\pi^{(F'')})_\alpha][\rho_2])_{F'',l'} = ([\rho_1],[\rho_2])_{F'',l'}.$$

Choose intermediate fields F_i', characters ψ_i, integers n_i, and elements $\sigma_{ijk} \in G_F$, as in item (8f) above. Write F_{ijk} for $(\sigma_{ijk}^{-1}F_i').F_j'$. Then

$$[r] = \sum_i n_i \mathrm{ind}_{F_i'/F}[r_{l,\imath}(\pi^{(F_i')} \otimes (\psi_i \circ \mathrm{Art}_{\,F_i'} \circ \det))]$$

in $\mathrm{Rep}_{F,l}$. This motivates us to set

$$A_{l',\imath',\alpha} = \sum_i n_i \mathrm{ind}_{F_i'/F}([r_{l',\imath'}(\pi^{(F_i')})_\alpha][(\imath')^{-1} \circ \psi_i]) \in \mathrm{Rep}_{F,l'}.$$

Note that

$$\dim A_{l',\imath',\alpha} = \sum_i n_i [F_i' : F] \dim r_{l',\imath'}(\pi)_\alpha = \dim r_{l',\imath'}(\pi)_\alpha.$$

Also note that

$$\begin{array}{rl} & (A_{l',\imath',\alpha}, A_{l',\imath',\alpha})_{F,l'} \\ = & \sum_{i,j,k} n_i n_j \\ & ([r_{l',\imath'}(\pi^{(F_{ijk})})_\alpha][(\imath')^{-1}\psi_i^{\sigma_{ijk}}|_{G_{F_{ijk}}}], [r_{l',\imath'}(\pi^{(F_{ijk})})_\alpha][(\imath')^{-1}\psi_j|_{G_{F_{ijk}}}])_{F_{ijk},l'} \\ = & \sum_{i,j,k} n_i n_j ([(\imath')^{-1}\psi_i^{\sigma_{ijk}}|_{G_{F_{ijk}}}], [(\imath')^{-1}\psi_j|_{G_{F_{ijk}}}])_{F_{ijk},l'} \\ = & (1,1)_{F,l'} \\ = & 1. \end{array}$$

Thus $A_{l',\imath',\alpha} = [r_{l',\imath',\alpha}]$ for some irreducible continuous representation $r_{l',\imath',\alpha}$ of G_F on a $\overline{\mathbb{Q}}_{l'}$-vector space of dimension $\dim r_{l',\imath'}(\pi)_\alpha$. Set

$$r_{l',\imath'} = r_{l',\imath',1} \oplus \cdots \oplus r_{l',\imath',t}.$$

We see that $r_{l,\imath} \cong r$ and that

$$\mathrm{tr}\, r_{l',\imath'}(\sigma) = \sum_i n_i \sum_{\tau \in G_F/G_{F_i'},\ \tau\sigma\tau^{-1} \in G_{F_i'}} ((\imath')^{-1}\psi_i(\tau\sigma\tau^{-1}))\mathrm{tr}\, r_{l',\imath'}(\pi^{(F_i')})(\tau\sigma\tau^{-1}).$$

Let v' denote a prime of F' and set $v = v'|_F$. By [Car10], if $v \nmid l'$ then

$$\imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F'_{v'}}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_{v'} \otimes |\det|_{v'}^{(1-n)/2})$$

is pure in the sense of [TY07]. (See the paragraph before Lemma 1.4 of that paper.) Hence $\imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})$ is also pure. (See parts (1) and (2) of Lemma 1.4 of [TY07].) Moreover if $\sigma \in W_{F_v}$ then

$$\begin{array}{rl} & \mathrm{tr}\,\imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})(\sigma) \\ = & \imath'\mathrm{tr}\,r_{l',\imath'}(\sigma) \\ = & \sum_i n_i \sum_{\tau \in G_F/G_{F_i'},\ \tau\sigma\tau^{-1} \in G_{F_i'}} \psi_i(\tau\sigma\tau^{-1})\imath'\mathrm{tr}\,r_{l',\imath'}(\pi^{(F_i')})(\tau\sigma\tau^{-1}) \\ = & \sum_i n_i \sum_{\tau \in G_F/G_{F_i'},\ \tau\sigma\tau^{-1} \in G_{F_i'}} \psi_i(\tau\sigma\tau^{-1})\mathrm{rec}(\pi^{(F_i')}_{\tau^{-1}v'|_{F_i'}})(\tau\sigma\tau^{-1}). \end{array}$$

(Again by the main theorem of [Car10].) If l'' is another prime with $v \not| l''$ and if $\imath'' : \overline{\mathbb{Q}}_{l''} \xrightarrow{\sim} \mathbb{C}$ then we conclude that

$$\imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})^{\mathrm{ss}} \cong \imath''\mathrm{WD}(r_{l'',\imath''}|_{G_{F_v}})^{\mathrm{ss}}.$$

As both are pure we conclude from part (4) of Lemma 1.4 of [TY07] that

$$\imath'\mathrm{WD}(r_{l',\imath'}|_{G_{F_v}})^{F-\mathrm{ss}} \cong \imath''\mathrm{WD}(r_{l'',\imath''}|_{G_{F_v}})^{F-\mathrm{ss}}.$$

Thus the $r_{l',\imath'}$ form a strictly pure compatible system. $\square$

If (π,χ) is a RAECSDC or RAESDC automorphic representation of $GL_n(\mathbb{A}_F)$ then we have remarked at the end of section 5.1 that $\{r_{l,\imath}(\pi)\}$ is a strictly pure compatible system of even weight w. Then $|\chi| = |\ |_{F^+}^{n-1-w}$. If π has central character χ_π then we see that $|\chi_\pi| = |\ |_F^{n(n-1-w)/2}$, and so $\pi \otimes |\det|_F^{(w+1-n)/2}$ has a unitary central character and so is unitary. If (π',χ') is a RAECSDC or RAESDC automorphic representation of $GL_{n'}(\mathbb{A}_F)$ and if $r_{l,\imath}(\pi')$ has weight $2(w'+1-n')$ and if S is a finite set of finite places of F then

$$\begin{array}{rl} & L^S(\pi \times (\pi')^\vee, s + (w - w' + n' - n)/2) \\ = & L^S((\pi|\det|_F^{(w+1-n)/2}) \times (\pi'|\det|_F^{(w'+1-n')/2})^\vee, s) \end{array}$$

is meromorphic and is holomorphic and non-zero at $s = 1$ unless

$$\pi \cong \pi'|\det|_F^{(w'-w+n-n')/2}$$

in which case it has a simple pole at $s=1$ (see [Sha81] and [JS81]).

Theorem 5.4.2. *Suppose that F is a CM (or totally real) field and that π is a RAECSDC (or RAESDC) automorphic representation of $GL_n(\mathbb{A}_F)$. If π has extremely regular weight, then there is a set of rational primes $\mathcal{L}$ of Dirichlet density 1 such that if $l \in \mathcal{L}$ and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ then $r_{l,\imath}(\pi)$ is irreducible.*

Proof. Let $\mathcal{L}$ be the set of rational primes of Dirichlet density 1 provided by Proposition 5.3.4 applied to the compatible system $\{r_{l,\imath}(\pi)\}$. Suppose $l \in \mathcal{L}$ and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let

$$r_{l,\imath}(\pi) = r_{l,\imath}(\pi)_1 \oplus \cdots \oplus r_{l,\imath}(\pi)_j$$

be a decomposition into irreducibles. Let F'/F and π_α for $\alpha = 1,\ldots,j$ be as in Proposition 5.3.4 for $\{r_{l',\imath'}(\pi)\}$ and $(l,\imath)$. Let S denote the finite set of primes of F which divide l or above which π ramifies or above which F' ramifies. Then

$$\mathrm{ord}_{s=1}\, L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s) = \mathrm{ord}_{s=1}\, L^S(\pi \times \pi^\vee, s) = -1.$$

We will show that $\mathrm{ord}_{s=1}\, L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s)$ also equals $-j$, and the theorem will follow.

Suppose that we are given an intermediate field $F' \supset F'' \supset F$ with F'/F'' soluble. By Lemma 1.4 (or 1.3) of [BLGHT09] there is a RAECSDC (or RAESDC) automorphic representation $\pi_\alpha^{(F'')}$ of $GL_{n_\alpha}(\mathbb{A}_{F''})$ such that $r_{l,\imath}(\pi_\alpha^{(F'')}) \cong r_{l,\imath}(\pi)_\alpha|_{G_{F''}}$. Moreover $r_{l,\imath}(\pi)_\alpha|_{G_{F''}}$ is irreducible. If $\psi : \mathrm{Gal}\,(F'/F'') \to \overline{\mathbb{Q}}_l^\times$ is a character then

$$\begin{array}{rl} & \mathrm{ord}_{s=1}\, L^S(\imath r_{l,\imath}(\pi)_\alpha|_{G_{F''}} \otimes r_{l,\imath}(\pi)_\beta^\vee|_{G_{F''}} \otimes \psi, s) \\ = & \mathrm{ord}_{s=1}\, L^S(\pi_\alpha^{(F'')} \times (\pi_\beta^{(F'')})^\vee \times (\imath \circ \psi \circ \mathrm{Art}\,_{F''}), s + (n_\beta - n_\alpha)/2) \\ = & -\delta_{\alpha,\beta}\delta_{\psi,1} \\ = & -([r_{l,\imath}(\pi)_\alpha|_{G_{F''}}][\psi], [r_{l,\imath}(\pi)_\beta|_{G_{F''}}])_{F'',l}, \end{array}$$

where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and equals 0 otherwise, and where $\delta_{\psi,1} = 1$ if $\psi = 1$ and equals 0 otherwise. [Note that the $\pi_\gamma^{(F'')}|\det|_{F''}^{(1-n_\gamma)/2}$ have different weights so that if $\pi_\gamma^{(F'')}|\det|_{F''}^{(1-n_\gamma)/2} \cong \pi_{\gamma'}^{(F'')}|\det|^{(1-n_{\gamma'})/2}(\imath \circ \psi \circ \mathrm{Art}\,_{F''} \circ \det)$ then $\gamma = \gamma'$. Moreover $r_{l,\imath}(\pi)_\gamma|_{G_{F''}} \cong r_{l,\imath}(\pi)_\gamma|_{G_{F''}} \otimes \psi$, and so, as $r_{l,\imath}(\pi)_\gamma|_{G_{F'}}$ is irreducible, $\psi = 1$. Similarly the $r_{l,\imath}(\pi)_\gamma|_{G_{F''}}$ have different Hodge-Tate numbers, so if $r_{l,\imath}(\pi)_\gamma|_{G_{F''}} \cong r_{l,\imath}(\pi)_{\gamma'}|_{G_{F''}} \otimes \psi$ then $\gamma = \gamma'$. Moreover as $r_{l,\imath}(\pi)_\gamma|_{G_{F'}}$ is irreducible we see that we also have $\psi = 1$.] Thus

$$\begin{array}{rl} & \mathrm{ord}_{s=1}\, L^S(\imath(r_{l,\imath}(\pi)|_{G_{F''}} \otimes r_{l,\imath}(\pi)|_{G_{F''}}^\vee \otimes \psi, s) \\ = & -(\mathrm{res}_{F''/F}[r_{l,\imath}(\pi)][\psi], \mathrm{res}_{F''/F}[r_{l,\imath}(\pi)])_{F'',l}. \end{array}$$

Now let F_i', n_i and ψ_i be as in part (8f) of the list at the start of this section. Then

$$L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s) = \prod_i L^S(\imath(r_{l,\imath}(\pi)|_{G_{F_i'}} \otimes r_{l,\imath}(\pi)|_{G_{F_i'}}^\vee \otimes \psi_i, s)^{n_i}$$

and

$$\begin{array}{rl} & \mathrm{ord}_{s=1}\, L^S(\imath(\mathcal{R} \otimes \mathcal{R}^\vee), s) \\ = & -\sum_i n_i([\psi_i]\mathrm{res}_{F_i'/F}[r_{l,\imath}(\pi)], \mathrm{res}_{F_i'/F}[r_{l,\imath}(\pi)])_{F_i',l} \\ = & -\sum_i n_i(\mathrm{ind}_{F_i'/F}([\psi_i]\mathrm{res}_{F_i'/F}[r_{l,\imath}(\pi)]), [r_{l,\imath}(\pi)])_{F,l} \\ = & -([r_{l,\imath}(\pi)], [r_{l,\imath}(\pi)])_{F,l} \\ = & -j, \end{array}$$

as desired. □

Theorem 5.4.3. *Suppose that F is a CM (or totally real) field and that $\mathcal{R}$ is a pure, totally odd, essentially conjugate self-dual, extremely regular, weakly compatible system of l-adic representations of G_F. Then we can write $\mathcal{R} = \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_s$ where each $\mathcal{R}_i$ is an irreducible, odd, essentially conjugate self-dual, strictly pure compatible system of l-adic representations of G_F.*

Proof. Choose a set $\mathcal{L}$ of rational primes of Dirichlet density 1 which simultaneously works for Propositions 5.2.2 and 5.3.4. (The intersection of two sets of Dirichlet density 1 has Dirichlet density 1.) Choose $\lambda | l \in \mathcal{L}$ such that l is unramified in F and $l \geq 2(n+1)$ and r_λ is crystalline with Hodge-Tate numbers all in an interval of the form $[a, a+l-2]$. Decompose r_λ into irreducible subrepresentations

$$r_\lambda = r_{\lambda,1} \oplus \cdots \oplus r_{\lambda,j_\lambda}.$$

By Theorem 5.4.1 each $r_{\lambda,\alpha}$ is part of a strictly pure compatible system $\mathcal{R}_\alpha$. Let F'/F and π_α for $\alpha = 1, \ldots, j_\alpha$ be as in Proposition 5.3.4 for $\mathcal{R}$ and λ. Then $\mathcal{R}_\alpha|_{G_{F'}}$ is the compatible system associated to π_α. Moreover π_α is extremely regular. By Theorem 5.4.2 there is a set $\mathcal{L}_\alpha$ of rational primes of Dirichlet density 1 such that if $\lambda' | l' \in \mathcal{L}_\alpha$ then $r_{\alpha,\lambda'}|_{G_{F'}}$ is irreducible. Thus $\mathcal{R}_\alpha$ is irreducible. □

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